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Reflecting on the 3x+1 Mystery. Outline of a Scenario - Improbable or Realistic ?

Abstract. Guessing the outcome of iterations of even most simple arithmetical functions could be an extremely hazardous experience. Not less harder, if at all possible, might be to prove the veracity of even a «sure» guess concerning iterations : this is the case of the famous *3x+1* conjecture. Our purpose here is to study and conceptualize some intuitive insights related to the ultimate (un)solvability of this conjecture.

The 3x+1 Conjecture

Guessing the outcome of iterations of even most simple arithmetical functions could be an extremely hazardous experience. Take, for example, the following elementary function defined by Lothar Collatz in 1932 [Lagarias 1985] :

$$\textbf{(1)} \qquad g(n) = \begin{cases} 2n/3 & \text{if } n \equiv 0 \bmod 3 \,; \\ 4n/3 - 1/3 & \text{if } n \equiv 1 \bmod 3 \,; \\ 4n/3 + 1/3 & \text{if } n \equiv 2 \bmod 3 \,. \end{cases}$$

It is easy to verify that *1* is the fixed point of g , $g(1) = 1$, and that the iterations of g starting at *2* and *4* yield the cycles of the length, respectively, two and five. On the other hand, it remains an open question whether the trajectory staring at *8* is divergent or ultimately cyclic. And yet, in the light of the unsolvability of Turing's halting problem, the impossibility to make even a plausible guess in a similar situation comes as no surprise to us.

It does surprise us, however, that not less harder, if at all possible, might be to prove the veracity of even a «sure» guess concerning iterations, that is, to prove the validity of an extremely simple and transparent outcome which is «forced upon us», so to say, by vast numerical checks and a host of plausible heuristic arguments. This is the case of the famous *3x+1* conjecture (its origins are traced to L. Collatz, too [Lagarias 1985]) :

The 3x+1 Conjecture. *Define the function T, acting on positive integers, as follows :*

$$\textbf{(2)} \qquad T : \mathbb{N} \to \mathbb{N} \;\; ; \;\; T(n) = \begin{cases} n/2 & \text{if } n \text{ is even} \,; \\ (3n+1)/2 & \text{if } n \text{ is odd} \,. \end{cases}$$

For any positive integer n, the sequence T(n), T(T(n)), ..., of iterates of T runs ultimately into the cycle $\{1 \to 2 \to 1\}$.

Since the problem became known about sixty years ago, many interesting and deep facts concerning the iterations of *T* have been discovered [Lagarias 1985], [Wirsching 1998], all confirming or otherwise «justifying» the *3x+1*-conjecture. However, no definite proof is as yet in sight, and the current state of affairs in the *3x+1* industry tends to confirm the old, startling remark of the late Paul Erdös : *«Mathematics is not yet ready for such problems»* (quoted in [Lagarias 1985]). One can only marvel at how such a straightforward and primitive in extremis iterative rule can produce such an immensely rich and mysteriously balanced dynamical pattern !

Remarkable Ramifications of Unsolvability

We attempt here to build on the subtle difference in the quality of our ignorance in the two examples above. Our purpose will be to informally conceptualize some intuitive insights related to the ultimate (un)solvability of the *3x+1* conjecture. To this end, we need to view the conjecture in a more general formal context, where unsolvability appears on a regular, well-understood basis. Then we should compare the emerging *3x+1* related «landscape» with similar unsolvability phenomena in other branches of mathematics.

In practice, this order will be reversed. Our first informal thesis is inspired by the theory of diophantine equations and Hilbert's tenth problem : we want to capture here, as if from an outer space, the mountainous diophantine landscape, with its lovely oases, its deserts, and its forbidding impenetrable peaks, - unsolvable diophantine equations. [1] Our second thesis (its very intuitive presentation will be made more formal in the following sections) carries out the comparison with the *3x+1* case. Then a proper technical context will be proposed.

As a matter of fact, the following statement spells out a well-known but not so well reported mathematical experience (cf., e. g., [Cherlin 1998], p. 1 [2]) :

Thesis 1. *The presence of unsolvability phenomena in a mathematical field brings with it the emergence of sometimes isolated and mutually almost unrelated «islands», - beautiful, deep, rich in consequences, and difficult «local» theories, with extremely complicated and / or lengthy proofs of the most important results. Or, for short, the proximity of unsolvability leads to* ***(i)*** *abrupt conceptual discontinuity,* ***(ii)*** *novel meaningfulness,* ***(iii)*** *startling logical complexity. Against such a background, the development of* ***(iv)*** *new powerful problem oriented methods is stimulated.*

An informal comparison of the diophantine and «iterative» landscapes could be carried out,

[1] Another question is what sort of a formal mechanism permits to discover and ultimately prove the unsolvability. As Robert Soare writes, «the frequent occurrence of r. e. <recursively enumerable> sets in <some> branches of mathematics and the existence of nonrecursive r. e. sets ... have yielded numerous undecidability results, such as the Davis-Matijasevich-Putnam-Robinson resolution of Hilbert's tenth problem on the solution of certain Diophantine equations, and the Boone-Novikov theorem on the unsolvability of the word problem for finitely presented groups.»

[2] « It will be seen that these two results, like other classification theorems, require a surprisingly lengthy argument, even though the classifications themselves are not very complex. It is not at all clear as yet why this is the case. The most obvious possibility is that our combinatorial tools need to be refined. A more intriguing possibility is that the classification of more general structures cannot be carried out effectively. This effectivity problem can be posed quite precisely, and is discussed in some detail ... below.»

as follows :

Thesis 2. *There is no doubt that the «unsolvability rank» of problems related to iterations of number-theoretical functions is, by their very iterative nature, much «higher» than, say, of diophantine problems. It means that the 3x+1 related landscape should be incomparably more «steep», irregular, unpredictable, and, as a rule, inaccessible than the diophantine one. In practice, the 3x+1 inspired research testifies in a most unambiguous way to at least three aforementioned phenomena : discontinuity, meaningfulness, complexity. Wanted : formal methods of noticeable power.*

Periodically Linear Functions

There is, however, in the last comparison an important gap : whereas the diophantine «landscape» is formed by all diophantine equations, no such a formal and existing *«by definition»* context is at hand in the *3x+1* case. At least if one disregards as too abstract and unnecessary powerful the full formalism of Turing's machines, with its theorem of unsolvability of the halting problem.

Fortunately, a more narrow framework, that of *periodically linear functions*, perfectly fits in with our purposes. Introduced by John Conway [Conway 1972], *periodically linear functions* represent a most natural and parsimonious generalization (more precisely, a straightforward parametrization) of the functions g and T :

Definition 1. Let $p \geq 2$ be an integer, and let

$$\boldsymbol{a} = (a_0 , \ldots , a_{p-1}) , \boldsymbol{b} = (b_0 , \ldots , b_{p-1}),$$

be a couple of p-tuples of rational numbers satisfying the following $2p$ conditions :

$$a_j p \in \mathbb{Z} , a_j j + b_j \in \mathbb{Z}, \text{ for } j = 0, \ldots , p-1 .$$

Then the couple $(\boldsymbol{a}, \boldsymbol{b})$ defines on the set $\mathbb{Z}$ a *periodically linear function* $\mathcal{T}_{\boldsymbol{a},\boldsymbol{b}}$ of integers, as follows :

$$\mathcal{T}_{\boldsymbol{a},\boldsymbol{b}}(n) = a_j n + b_j , \text{ if } n \equiv j \bmod p .$$

(Choosing, in particular,

$$p = 3 , \boldsymbol{a} = (2/3 , 4/3 , 4/3) , \boldsymbol{b} = (0 , -1/3 , 1/3) ,$$
$$p = 2 , \boldsymbol{a} = (1/2 , 3/2) , \boldsymbol{b} = (0 , 1/2) ,$$

one gets, respectively, the functions g and T .)

J. Conway has also explicitly constructed a periodically linear function h, acting, as T, on positive integers, $h: \mathbb{N} \to \mathbb{N}$, with the iterates encoding Turing's halting problem. Thus, the «periodically linear landscape» is rich enough to naturally accommodate both the *3x+1* case and some explicit unsolvability phenomena.

Note. (1) As a matter of fact, Conway's construction does not use the full strength of the above definition : his are *homogeneous* periodically linear functions,

$g = \mathcal{T}_{\boldsymbol{a},\boldsymbol{0}}$, with $\boldsymbol{a} = (a_0 , \ldots , a_{p-1})$, $a_j > 0$, for $j = 0, \ldots , p-1$, and $\boldsymbol{b} = \boldsymbol{0} = (0, \ldots, 0)$,

whose iterates are repeated multiplications by rational numbers a_j, followed by mod p reductions :

$$g^{(k)}(n) = (a_{j_k} \cdot (a_{j_{k-1}} \cdot \ ...(a_{j_2} \cdot ((a_{j_1} \cdot n) \bmod p) \bmod p) ... \ ) \bmod p) \bmod p,$$

(3) where

$$0 \le j_1, ..., \ j_k \le p\text{-}1 \ , j_1 \equiv n \bmod p, j_2 \equiv a_{j_1}.n \bmod p \ , ... \ , \ j_k \equiv a_{j_{k-1}}.n \bmod p.$$

Imitating Kurt Gödel's classical number-theoretical coding technique used in the proof of Gödel's incompleteness theorem, Conway has shown that, for any partial recursive function f (including that which encodes Turing's halting problem), a proper choice of parameters p and $a_0 \ , \ ... \ , \ a_{p-1}$ guarantees that the interplay (3) between iterates of the corresponding g simulate f, in the following sense (see for details [Conway 1972], [Lagarias 1985]) :

(a) $g^{(k)}(n) = 2^s$, for some $k \ge 1$ and some s, if and only if f is defined on n;
(b) $g^{(k)}(n) = 2^{f(n)}$, for the minimal $k \ge 1$ such that $g(n)$ is a power of 2.

(2) In some cases, the condition that the function $\mathcal{T}$ acts on the full set $\mathbb{Z}$ of integers is too restrictive [Belaga 1995]. It can be weakened in many ways. A generalisation of the above definition to *piecewise* periodically linear functions acting on integers not divisible by a nontrivial divisor d of the *period* p can be found in the Annex below.

The 3x+1 Paradox

Conway's discovery confirms the difficulty of both Collatz's questions : predicting the behaviour of the trajectory $g^{(k)}(8)$ and proving the *3x+1 conjecture* might well be unsolvable problems.

Unfortunately, such a distant unsolvability prospect sheds not much light on the *3x+1* case, flying in the face, so to say, of the remarkable wealth of its experimental and heuristic confirmations.

The conflict (already anticipated in our Thesis 2) is a sign, as the present author is convinced, that

Thesis 3. *We are confronted here with not just a difficult question about terminations of iterates, but with a paradox of a new type, for which there exists as yet no proper formal discourse.*

In what follows, we shall try to make this vague assertion as precise as possible. Let us start with a tentative informal description of the paradox in question, with the term *paradox* understood in its simple original sense, i. e., as a conjunction of two separately plausible and mutually incompatible statements :

The 3x+1 Conjectural Paradox. (i) *It might be well that the evidence in the favour of unsolvability of the 3x+1 conjecture would steadily grow until «almost» become a certainty.* ***(ii)*** *It also might be that all direct investigations and calculations would confirm the conjecture with ever growing evidence.*

Or, in a more formal way : Let $T^*(1) \subseteq \mathbb{N}$ be the set of all iterated inverse images $T^{(-k)}(1)$ of 1, i. e., the set of all positive integers n, whose T-iterates land ultimately at 1, and then run the cycle $\{1 \to 2 \to 1\}$. The set $T^*(1)$ is recursively enumerable; it is enumerated by repeated recursive applications, starting with $T^{(-1)}(1) = \{2\}$, of the inversion formula :

(4) $T^{(-1)}(n) = \{(2n\text{-}1)/3, \ 2n\}$, if $2n \equiv 1 \bmod 3$; $\{2n\}$, otherwise .

A proof that the complement $\mathbb{N} \setminus T^*(1)$ is not recursively enumerable would imply unsolvability of the *3x+1* conjecture (cf. Note 2 above). Even in the absence of a definitive proof, the reasons to believe $\mathbb{N} \setminus T^*(1)$ is not recursively enumerable (and thus, the conjecture is unsolvable) might become overwhelming.

On the other hand, ever widening evidence might indicate that this complement is empty, $\mathbb{N} \setminus T^*(1) = \emptyset$, thus, *ipso facto*, confirming the conjecture.

The paradox can be also reformulated and specified in terms of «fast growing» functions, - see below Thesis 4 and the related deliberations. In either way, the possible, - and desirable, - outcome of these two developing and conflicting mathematical accounts should be an eventual deepening of our understanding of what constitutes a *mathematical proof*, - beyond the acknowledged now limits of *provability*. (This might be one of the possible interpretations of the aforementioned remark of P. Erdös : *«Mathematics is not yet ready for such problems»*).

In the absence of such a new understanding, one should proceed both ways, «*negative*» and «*positive*» : trying to confirm the conjecture, without losing the sight on the efforts of proving its unsolvability.

Methodological Aside. The vagueness of the above wording of our aspirations («deepening of our understanding of what constitutes a *mathematical proof*, - beyond the acknowledged now limits of *provability*») is intentional. We talk here about the *basic paradigm of mathematical proof*, of which, say, the axiomatic aspect is only a part (albeit, one of the most important, - cf. Thesis 4 below ; still more important might be the universal cultural «abstraction ascent»). Clearly, the Greeks didn't invent non-Euclidian geometry (and would have a great difficulty to understand it) just because they were not aware of the independence of the V-th postulate from other (known to them) axioms of Euclidian geometry. [3]

Widening the 3x+1 Case to Its 3x+d Analogue

The unsolvability case being discussed above, let us try now the other way. There follows a brief description of a simple generalization of the conjecture which permits to better appreciate its particularity, beauty, and complexity ; it also shows that the formalism of periodically linear functions can work both ways. Besides, the generalization will supply us with few hints on how might evolve the interpretation of the above paradox.

We widen the *3x+1* definition (2) by introducing one free parameter [Lagarias 1990], [Belaga, Mignotte 1998] :

for any odd $d \geq -1$, $d \not\equiv 0 \bmod 3$,

(5) T_d: $\mathbb{N} \to \mathbb{N}$; $T_d(n) = n/2$ if n is even ; $(3n+d)/2$ if n is odd .

[3] A possible future paradigm shift in axiomatics is discussed in [Friedman 1992], p. 51 : «The fundamental issue is this : Is there a basic mathematical problem about standard finite objects such as, say, natural numbers or rational numbers ..., etc., with a clear and intuitive meaning, conveying interesting mathematical information, that is readily graspable, and which is independent of ZFC ? We speculate that sometime during the twenty-first century, someone will answer the above question in affirmative ... » Restricting ourselves to this axiomatic aspect, we submit below that the *3x+1* conjecture is independent of the second order arithmetics Z_2 .

Note that $\{d \to 2d \to d\}$, call it the *trivial*, or *basic d-cycle*, plays for for the iterates of T_d the same role as $\{1 \to 2 \to 1\}$ plays for the iterates of $T = T_1$. However, here the similarity ends, and the high balancing act of the *3x+1* case breaks down :

Calculations show that, for any $d \neq 1$ defined as in (5), the function T_d ***always*** *has at least few nontrivial cycles* [4]*. For example, T_{13} has at least* ***9*** *such cycles, of the lengths, respectively, 4 (1 cycle), 8 (7), and 24 (1); T_{71} has* ***7****, 10 (2), 27 (5); T_{91} has* ***13****, 4 (2), 8 (7), 12 (2), 24 (1), 48 (1) .*

Thus, it is impossible to literally transpose the *3x+1* conjecture into the *3x+d* context. Still, calculations show that, for a given *d*, the number of different cycles is finite, and that «big» *3x+d* trajectories finally «collapse», as do their *3x+1* analogues. The following reformulation of the *3x+1* conjecture better fits this new dynamics :

The 3x+1 Conjecture*.
(1) *There are no divergent trajectories T(n), T(T(n)), ..., of iterates of T.*
(2) *The only cycle is that of $\{1 \to 2 \to 1\}$.*

The generalization to the *3x+d* case : change (2) to «*The number of cycles is finite*» [Lagarias 1990]. Or state it in the following «collapsing» way [Belaga, Mignotte 1998] :

The 3x+d Conjecture. *For any odd and not divisible by 3 integer $d \geq -1$, there exists such an integer $N_d > 0$ that, for any positive integer n, the sequence $T_d(n)$, $T_d(T_d(n))$, ..., of iterates of T_d ultimately collapses «under» N_d . More precisely, for any $n > 0$, there exists such an integer $K_{d,n} > 0$ that, for all $k > K_{d,n}$, $T_d^{(k)}(n) \leq N_d$.*

This generalization illustrates how subtle, unique, and, apparently, extremely difficult (and possibly, *improvable*!) is the condition of the *3x+1* conjecture demanding that *all the iterates of T run ultimately into the single cycle $\{1 \to 2 \to 1\}$.*

On the other hand, the experimentally confirmed plausibility of the existence of a «collapsing mechanism», common to all *3x+d* cases, holds out some hope for a future proof of the «divergent» part of the *3x+1* conjecture.

Still, the availability of such a proof could be impeded by «fast growing» obstacles, to the description of which we now proceed.

Fast Growing Functions and Termination Proofs

The unsolvability of Turing's halting problem does not block, of course, resolution of some individual termination problems. Thus, for example, Reuben L. Goodstein [Goodstein 1944] has constructed a truly elementary function $n \to g(n)$, definable in Peano arithmetics, whose iterates $g^k(n)$ ultimately terminate at *0* for any *n*. However, with *n* growing, it takes them very long indeed to arrive at *0*, which means that the function

$$K(n) = min\ (k,\ g_k(n) = 0)$$

is growing so fast, that any proof of this fact necessary uses a mathematical induction

[4] More precisely, *primitive* cycles, - see below. [Lagarias 1990] conjectures that the number of primitive cycles is positive for any *d*.

through transfinite numbers up to ε_0 , and, thus, cannot be carried out in Peano arithmetics [Kirby, Paris 1982] !

Also, Harvey Friedman discovered a remarkably transparent combinatorial theorem which (very roughly) states that in any sufficiently long *finite* sequence of finite trees, there is a pair of trees, with one tree embeddable into the second one [Harrington, Morley, Scedrov, Simpson 1985]. (This is a *finitistic* version, called *FFF*, of Kruskal's classical theorem, dealing in its original form with *infinite* sequences of finite trees). Friedman's construction is elementary and *predicative*. However, the proof of *FFF demonstrably* requires mathematical induction up to the first *impredicative denumerable ordinal* Γ_0 [Gallier 1991].

Those and similar phenomena [Simpson 1987] confirm (or, if one prefers, make «axiomatic sense» of) our Theses 2 and 3, opening the way for their formalization, as follows :

Thesis 4. *In many interesting cases, the termination can be proved only in the axiomatic domain which vastly, and unpredictably, outstrips the original domain to which the problem has been directly related.*

Going now back to *3x+1* and *3x+d* iterations, take the above «collapsing» formulation of the *3x+d* conjecture and. It asserts that any initially divergent trajectory would finally collapse into some (possibly, growing very fast with *d*) N_d-vicinity of *1*.

Now, among possible scenarios of such a collapse, the following one, being quite realistic, would create serious difficulties for any (imaginable at present) proof :

Suppose there exists such a *fast growing* sequence of integers $n_{d,q}$, $q \geq 1$, that the T_d-trajectories starting at $n = n_{d,q}$ would begin their N_d-descent only after a very long ascent, with the ascent span $K_{d,n}$ growing very fast with *d* and *n*.

In other words, the collapse in question might depend quantitatively on *so fast growing functions*

$$N_d = N(d),\ n_{d,q} = n(d,q)\ ,\ \text{and}\ K_{d,n} = K(d,n)\ ,$$

that its proof would require axiomatic means vastly outstripping those available now in Mathematics (technically, one talks here about the second order arithmetics Z_2). - Which would bring us back to the above *3x+1* paradox !

Primitive Cycles and Fast Growing Factors of Numbers $2^p - 3^k$

Another outstanding conjecture related to *3x+d* iterations stumbles apparently on a similar fast growing obstacle.

Jeff Lagarias has defined a T_d-cycle to be *primitive* if the greatest common divisor of its members is *1* ; he also conjectured that, *for any odd and not divisible by 3 integer $d \geq -1$,* there exists a primitive T_d-cycle [Lagarias 1990]. The conjecture is verified for all $d \leq 1000$ [Belaga, Mignotte 1999].

However, both theoretical considerations and computations show that, to be true, the

conjecture implies an occasional but persisting appearance of very long primitive T_d-cycles whose existence depends on a huge common factor (the number C below) of a pair of «very big» numbers (A and B) related to the structure of the cycle.

More precisely, let p be the length of a T_d-cycle, k be the number of odd members of the cycle, n be its first and minimal member (always odd), and let

$$p_j \ , \ 1 \le j \le k\text{-}1 \ , \ 0 < p_1 < p_2 \ldots < p_{k-1} < p$$

be the index of the $(j+1)$-th odd member of the cycle [Belaga, Mignotte 1999]. One associates with the cycle two «big» positive integers, A and B , as follows

$$A = 2^p - 3^k \ , \ B = 3^{k-1} + 3^{k-2}.2^{p_1} + 3^{k-3}.2^{p_2} + \ \ldots \ + \ 3.2^{p_{k-2}} \ + \ 2^{p_{k-1}},$$

and one defines $C = gcd\,(A, B)$.

Apparently, there are no *a priori* reasons to believe that the two numbers A and B should have a «big» common divisor C. (As a matter of fact, the arithmetics of the above numbers is extremely obscure [Wirsching 1998]).

However, it is easy to verify that the numbers A, B, C are related to «small» parameters d and n of the cycle by the formulae [Lagarias 1990], [Belaga, Mignotte 1998] :

$$d = A/C \ , \ \ n = B/C \ .$$

Which implies that C needs to be a «very big» number, too !

Here again, for a rarefied (and, thus, *fast growing*) sub-sequence of integers d, the related growth of the corresponding «primitive» parameters p, k, and n could induce such a «fast» growth of the numbers A , B , C, that the aforementioned phenomenon of «fast growing» improvability might inescapably emerge.

Annex : Piecewise Periodically Linear Functions

In many cases, the condition that a periodically linear function acts on the full set $\mathbb{Z}$ of integers is too restrictive. We generalize here Definition 1 to *piecewise* periodically linear functions acting on integers not divisible by a nontrivial divisor d of the *period* p :

Definition 2. Let $p \ge 2$ be an integer and $d \ge 2$ be its divisor. Let $\mathbb{Z}_{(d)}$ be the set of integers relatively prime to d, and let $\{ \ r_0 \ , \ r_1 \ , \ \ldots \ , \ r_{s-1} \ \}$ be the subset of positive integers «cut off» from $\mathbb{Z}_{(d)}$ by the period p,

$$1 \le r_0 < r_1 < \ \ldots \ < r_{s-1} \le \ p\text{-}1 \ ; \ \ r_j \not\equiv 0 \bmod d, \ \ 0 \le j \le s\text{-}1 \ \ ; \ \ \ s = p\phi\,(d)/d \ ,$$

where $\phi\,(d)$ is Euler's function. Finally, let

$$\boldsymbol{a} = (a_0 \ , \ \ldots \ , \ a_{s-1}) \ , \ \boldsymbol{b} = (b_0 \ , \ \ldots \ , \ b_{s-1}),$$

be a couple of s-tuples of rational numbers satisfying the following $2s \ = \ 2p\phi\,(d)/d$ conditions :

(6) $\qquad a_j\, p/d \ \in \mathbb{Z} \ , \quad a_j\, r_j + b_j \in \mathbb{Z}_{(d)}, \quad$ for all $j, \ 0 \le j \le s\text{-}1$.

Then the parameters p, d, $\boldsymbol{a}$, $\boldsymbol{b}$ define a *piecewise periodically linear* (*pwpl*-) *function* $\mathcal{T}_{p,d,\boldsymbol{a},\boldsymbol{b}}$, as follows :

$\mathcal{T}_{p,d,\boldsymbol{a},\boldsymbol{b}} : \mathbb{Z}_{(d)} \rightarrow \mathbb{Z}_{(d)}$; $\mathcal{T}_{p,d,\boldsymbol{a},\boldsymbol{b}}(n) = a_j\, n + b_j$, if $n \in \mathbb{Z}_{(d)}$ and $n \equiv r_j \mod p$.

Remark. **(1)** To verify that $\mathcal{T}_{p,d,\boldsymbol{a},\boldsymbol{b}}$ sends an integer not divisible by d to another such integer, observe that, for any $n \in \mathbb{Z}_{(d)}$, there exists a unique j, $0 \le j \le s-1$, such that $n \equiv r_j \mod p$; thus, according to (6), for any such n ,

$$\mathcal{T}_{p,d,\boldsymbol{a},\boldsymbol{b}}(n) = a_j\, n + b_j = a_j \,.\, (r_j + m \,.\, p) + b_j = (a_j \,.\, r_j + b_j) + a_j.m.p \in \mathbb{Z}_{(d)} .$$

(2) The actual number of parameters $(a_j\ ,\ b_j)$ does not need to be as big as $2s$. Thus, [Belaga 1995] (cf. also [Belaga, Mignotte 1998]) defines a *pwpl*-function W acting in a two-to-one way on the set $\mathbb{N}_{(6)}$ of positive integers not divisible by 6. The function W is closely related to the $3x+1$ transform T , «correcting» its unpleasant property to be an [either-two-or-one]-to-one function : cf. the formula (4) above. W is formally characterized by the parameters $p = 96$, $d = 6$, $2s = 64$, but is actually defined by 12 rational numbers on 8 classes *mod* $12, 24, 48, 96$.

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