Problem Set 1

1. Determine the final digit of
$$\sigma_n = \sum_{k=1}^{n} k$$
for each positive integer n. Produce a proof.

2. Determine the limits
$$\liminf_{x\to\infty}(1+\cos(x)/x)^x$$
$$\limsup_{x\to\infty}(1+\cos(x)/x)^x$$
Produce a proof.

3. Given a monic polynomial
$$x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0,$$
express the sum of the squares of the roots in terms of the coefficients. Produce a proof.

4. Find the absolute value of each of the roots of $11z^{10} + 10iz^9 + 10iz - 11$. No proof needed.

5. Identify the products
$$\prod_{r=1}^{\lfloor (n-1)/2 \rfloor} \left(-1 - 2\cos\frac{2\pi r}{n}\right)$$
and
$$\prod_{r=1}^{\lfloor (n-1)/2 \rfloor} \left(3 + 2\cos\frac{2\pi r}{n}\right)$$
for each positive integer n. No proof needed.

6. A standard calculus problem considers a long vertical strip of paper of width w. The paper is folded so that the upper left corner touches the right edge; see Figure 1. The goal is to determine the specific fold so that the left diagonal line is minimized.

   (a) Show that
   $$y^2 = \frac{2x^3}{2x - w}.$$

   (b) Use calculus within Maple to find a candidate for the optimal choice of x. Prove that this choice is indeed optimal.

   (c) Use algebra within Maple (no calculus) to prove that your candidate from the last part is optimal for the problem. Hint: If x^* is the optimal value and $f(x) = 2x^3/(2x - w)$, consider the expression $f(x) - f(x^*)$.

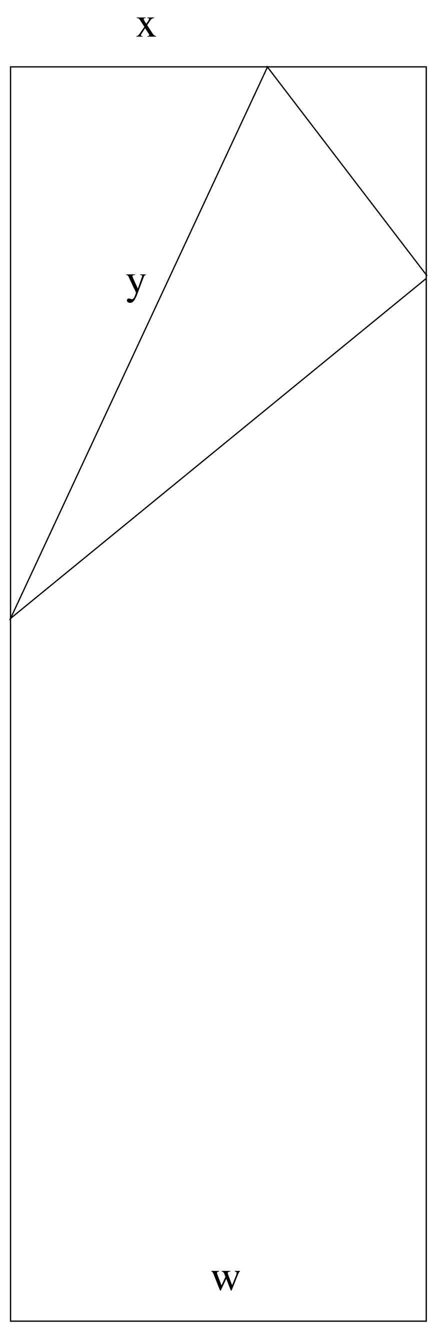

Figure 1: The folded paper problem.