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CHARACTERIZING ASYMPTOTIC STABILITY WITH DULAC FUNCTIONS

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Abstract. This paper studies questions regarding the local and global asymptotic stability of analytic autonomous ordinary differential equations in $\mathbb{R}^n$. It is well-known that such stability can be characterized in terms of Liapunov functions. The authors prove similar results for the more geometrically motivated Dulac functions. In particular it holds that any analytic autonomous ordinary differential equation having a critical point which is a global attractor admits a Dulac function. These results can be used to give criteria of global attraction in two-dimensional systems.

1. Introduction and Main Results.

This paper has two themes: local asymptotic stability and global asymptotic stability. We state our main results and some preliminary definitions in the following two subsections.

1.1. Local asymptotic stability.

This part of the paper is devoted to the relationship between asymptotic stability and the existence of Dulac functions. We formulate the following question: *Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be an analytic vector field and suppose that the system* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ *has a critical point* $\mathbf{p}$ *which is locally asymptotically stable. Let* $N = W^s(\mathbf{p})$ *denote its basin of attraction. Does there exists a smooth function* $B : N \to \mathbb{R}^+$*, usually called a Dulac function, such that* $\operatorname{div}(B\mathbf{F})$ *is always negative?* The question is formalized below and proofs to affirmative results are given in Section 2. In particular an affirmative answer to this question would allow one to restrict the study of the global asymptotic stability to vector fields with negative divergence. The question is also related to the study of sufficient conditions which guarantee global asymptotic stability. In particular, Section 3 details connections between such sufficiency results and Dulac functions when $n = 2$.

In order to state our results relating asymptotic stability with the existence of Dulac functions, we begin with some definitions and preliminary results. Consider

$$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \,,\; \mathbf{x} \in \mathbb{R}^n \tag{1}$$

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where $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is C^∞ or analytic. We say that the critical point $\mathbf{p}$ is an **attractor point** of (1) if $\varphi(t, q)$ tends to $\mathbf{p}$ as t tends to infinity for all $\mathbf{q}$ in some open set containing p, where $\varphi(t, \mathbf{q})$ denotes the solution of (1) under the initial condition $\varphi(0, \mathbf{q}) = \mathbf{q}$. The maximal open set N of points $\mathbf{q}$ satisfying the above condition is called the **basin of attraction** of $\mathbf{p}$. Note that N is the stable manifold of $\mathbf{p}$, i.e. $N = W^s(\mathbf{p})$. In the case $N = \mathbb{R}^n$, we say that $\mathbf{p}$ is a **global attractor.** System (1) is said to have **local asymptotic stability (LAS)** if it has an attractor point $\mathbf{p}$ which is stable (in the Liapunov sense) and that system (1) has **global asymptotic stability (GAS)** if it has **LAS** and the attractor point is global.

We will mainly use the result (see [21]) which gives the characterization of LAS via Liapunov functions, which asserts that (1) admits LAS at $\mathbf{p}$ if and only if there exists a function $f : N \to \mathbb{R}$ such that $f(\mathbf{x}) > 0$ for all $x \in N \setminus \{\mathbf{p}\}$, $f(\mathbf{p}) = 0$ and $\dot{f}(\mathbf{x}) := \sum_{i=1}^n \frac{\partial f_i}{\partial x_i}(\mathbf{x}) F_i(\mathbf{x}) < 0$ for all $\mathbf{x} \in N$, $\mathbf{x} \neq \mathbf{p}$. Furthermore, the level sets of f, i. e., $\{\mathbf{x} : f(\mathbf{x}) = h\}$ are compact sets and

$$\lim_{\mathbf{x} \to \partial N} f(\mathbf{x}) = \infty,$$

where ∂N denotes the boundary of N. Concerning regularity, we know that f is of class $\mathcal{C}^\infty$ if $\mathbf{F}$ is also $\mathcal{C}^\infty$; see [20]. We remark that having $\mathbf{F}$ analytic does not necessarily imply that f is also analytic, as Examples 3 and 4 show. Finally, if $\Omega \subset \mathbb{R}^n$ is open and $B : \Omega \longrightarrow \mathbb{R}$ is a C^1 function, we say that B is a **Dulac function** on Ω of system (1) if $B(\mathbf{x}) \geq 0$, for all $\mathbf{x} \in \Omega$, $B(\mathbf{x}) = 0$ implies $\mathbf{F}(\mathbf{x}) = 0$ and $\operatorname{div}(B\mathbf{F})(\mathbf{x}) \leq 0$ for all $\mathbf{x} \in \Omega$, recalling that $\operatorname{div} \mathbf{F}(\mathbf{x}) = \sum_{i=1}^n \frac{\partial F_i}{\partial x_i}(\mathbf{x})$. When B is a Dulac function and there is a compact subset K of Ω and a positive constant k such that $\operatorname{div}(B\mathbf{F}) \leq -k < 0$ in $\Omega \setminus K$, we will say that B is a **strong Dulac function**. Let $\varphi(t, \mathbf{x})$ denote the solution of the equation $\dot{\mathbf{x}} = B(\mathbf{x})\mathbf{F}(\mathbf{x})$ with initial condition $\mathbf{x}$. Note that if Ω is invariant under this flow and B is a Dulac function on Ω, then for all positive t the map $\mathbf{x} \longrightarrow \varphi(t, \mathbf{x})$ decreases any finite volume in Ω.

We formulate the following

Question. *Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be an analytic vector field and suppose that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ has LAS at p. Is it true that there exists a C^∞ Dulac function B for system (1) defined in $W^s(\mathbf{p})$?* If the answer is yes, our next question is: *Can this B be chosen such that it is a strong Dulac function?* Notice that the geometric interpretation of a positive answer to this second question is that the flow can be reparametrized in such a way so that, far from the singularity, the dissipation of the volume is in some way uniformly contracting. The existence of a strong Dulac function, even in the plane, is not sufficient for a system having LAS to be GAS; see Example 6 for a polynomial example or [6] for an analytic one. Notice also that the Lorenz system provides a polynomial example in $\mathbb{R}^3$ with many periodic orbits, complex behavior and a strong Dulac function ($B(x) \equiv 1$). As we will see in Section 3, the property of having a strong Dulac function, together with other conditions on the system, can be used to study GAS in the plane, see Theorems 5 and 6.

Our main results concerning the above questions are the following three Theorems, which are proved in Section 2.

Theorem 1. *Let $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ vector field and suppose that system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ has LAS at $\mathbf{p}$ and $div\mathbf{F}(\mathbf{p}) < 0$. Then there exists a strong Dulac function B defined in $W^s(\mathbf{p})$. Moreover this function B can also be chosen such that $\operatorname{div}(B\mathbf{F})(\mathbf{x})) < -k$ for some $k > 0$ and for all $\mathbf{x} \in W^s(\mathbf{p})$. Furthermore, B can be taken analytic if $\mathbf{p}$ admits an analytic Liapunov function.*

Note that the above result gives an affirmative answer to our question in the hyperbolic case. For the non-hyperbolic case, we only give partial results which are summarized in the following Theorem.

Theorem 2. *Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be a smooth vector field and suppose that system* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ *has LAS at* $\mathbf{p}$ *with* $\operatorname{div}\mathbf{F}(\mathbf{p}) = 0$. *Assume that either* $\mathbf{F}$ *is* $\mathcal{C}^\infty$ *and* $\operatorname{div}\mathbf{F}(\mathbf{x}) \leq 0$ *in a neighbourhood of* $\mathbf{p}$ *or that* $\mathbf{F}$ *is analytic and* $\mathbf{p}$ *admits an analytic Liapunov function. Then there exists a* C^∞ *strong Dulac function defined in* $W^s(\mathbf{p})$.

As we will see, if $\mathbf{F}$ admits LAS at $\mathbf{p}$, then either there exists a neighbourhood of $\mathbf{p}$ in which $\operatorname{div}\mathbf{F}$ is non-positive or in any neighbourhood of $\mathbf{p}$ the divergence changes sign. Example 1 exhibits this last situation.

Lastly, we present a result which shows that the only obstruction to obtaining a positive answer to our question is the smoothness of the Dulac function at $\mathbf{p}$.

Theorem 3. *Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be a* C^∞ *vector field and suppose that the system* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ *admits LAS at* $\mathbf{p}$. *Then there exists a continuous function* $B : W^s(\mathbf{p}) \longrightarrow \mathbb{R}$ *which is of class* C^∞ *in* $W^s(\mathbf{p}) \setminus \{\mathbf{p}\}$ *such that* $\operatorname{div}(B\mathbf{F})(\mathbf{x}) < 0$ *for all* $\mathbf{x} \in W^s(\mathbf{p}) \setminus \{\mathbf{p}\}$. *Furthermore, this Dulac function* B *can also be chosen as a strong Dulac function.*

1.2. **Global asymptotic stability.** As a converse to the theorems stated in the above section which give necessary conditions for the existence of an appropriate Dulac function, Section 3 studies what additional conditions may arise to give sufficient conditions for global asymptotic stability in the plane.

Before giving the main results concerning GAS, we introduce a new definition for a C^1 vector field $\mathbf{F} : \mathbb{R}^\mathbf{2} \longrightarrow \mathbb{R}^\mathbf{2}$. Given a point $\mathbf{q} \in \mathbb{R}^\mathbf{2}$ we will denote the positive (respectively negative) semi-orbit beginning at $\mathbf{q}$ by $\gamma_+(\mathbf{q})$ (respectively $\gamma_-(\mathbf{q})$). Also we denote be $\gamma(\mathbf{q})$ the complete orbit of $\mathbf{q}$. A **saddle at infinity** (SAI) is a pair of semi-orbits $\gamma_+(\mathbf{q}),\ \gamma_-(\mathbf{r})$ satisfying the following three conditions:

- The ω-limit of $\mathbf{q}$ and the α-limit of $\mathbf{r}$ are empty.
- There exist one-side compact transversal sections Σ_+ at $\mathbf{q}$ and Σ_- at $\mathbf{r}$ such that a Poincaré map $\pi : \Sigma_+ \setminus \mathbf{q} \longrightarrow \boldsymbol{\Sigma}_- \setminus \mathbf{r}$ may be defined and satisfies
$$\lim_{\mathbf{x}\to\mathbf{q}} \pi(\mathbf{x}) = \mathbf{r}.$$
- If $\gamma(\mathbf{r}) = \gamma(\mathbf{q})$ then $\mathbf{q} \in \gamma_+(\mathbf{r})$ and $\mathbf{q} \neq \mathbf{r}$

We call each of the semi-orbits the separatrices of the SAI. We stress the fact that $\gamma(\mathbf{q})$ and $\gamma(\mathbf{r})$ could be the same orbit (see the right hand picture of Figure 1). In this case the third condition in the definition is essential to ensure that the orbits near $\mathbf{q}$ escape to infinity before returns near $\mathbf{q}$. The additional condition that $\mathbf{q} \neq \mathbf{r}$ is not essential but it allows for a unified notation. Note that the first condition ensures that $\gamma_+(\mathbf{q})$ and $\gamma_-(\mathbf{r})$ escape to infinity. Note also that when $\mathbf{F}$ can be compactified, this situation implies the existence of at least one hyperbolic sector associated with a critical point at infinity. This notion is already introduced in some works (see for instance [14] or [16, p. 409]). Finally, note that in the case $\gamma(\mathbf{q}) \neq \gamma(\mathbf{r})$, a SAI gives rise to an unbounded Reeb component.

Theorem 4. *Let* $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *vector field having a unique critical point* $\mathbf{p}$ *which is locally asymptotically stable. Assume that* $\mathbf{F}$ *admits a Dulac function (or alternatively that it has no periodic orbits). Then* $\mathbf{p}$ *is globally asymptotically stable if and only if* $\mathbf{F}$ *has no saddles at infinity.*

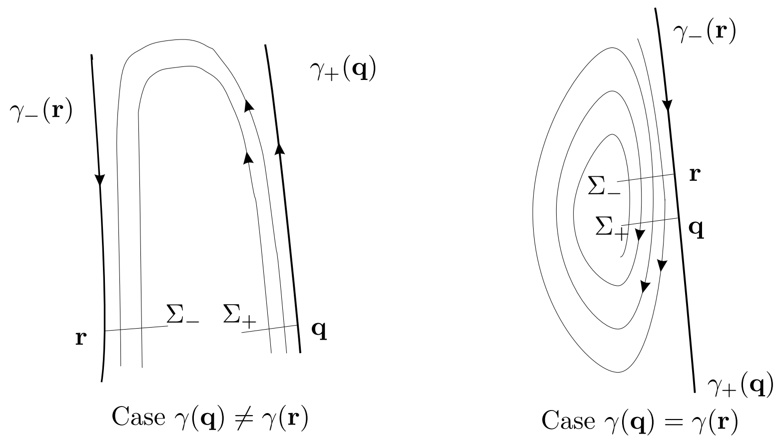

FIGURE 1. Examples of saddles at infinity (SAI).

By using the above theorem we get some results which guarantee GAS.

Theorem 5. *Let* $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *vector field having a unique critical point* $\mathbf{p}$ *which is locally asymptotically stable. Assume that* $\mathbf{F}$ *admits a strong Dulac function and that one of the following conditions holds:*

A) The vector field $\mathbf{F}$ *is polynomial and the critical points at infinity of its Poincaré compactification are elementary.*

B) The absolute value of the curvature of orbits of $\mathbf{F}$ *is bounded in a neighbourhood of infinity.*

Then $\mathbf{p}$ *is globally asymptotically stable.*

Remark 1. Condition (A) of the above theorem is easily verifiable. Let n be the degree of $\mathbf{F}$ and P_n and Q_n the highest degree terms of its components. Define $R_n(\theta)$ and $S_n(\theta)$ as

$$R_n(\theta) = \cos\theta\, Q_n(\cos\theta, \sin\theta) - \sin\theta\, P_n(\cos\theta, \sin\theta),$$
$$S_n(\theta) = \cos\theta\, P_n(\cos\theta, \sin\theta) + \sin\theta\, Q_n(\cos\theta, \sin\theta).$$

Condition (A) is equivalent to: For each θ^* satisfying $R_n(\theta^*) = 0$ we have

$$(R_n'(\theta^*))^2 + (S_n(\theta^*))^2 \neq 0.$$

The above Theorem can also be reformulated as follows:

Theorem 6. *Let* $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ *be a* C^1 *vector field having a unique critical point* $\mathbf{p}$ *which is locally asymptotically stable and assume that either condition (A) or Condition (B) of Theorem 5 holds. Then* $\mathbf{p}$ *is GAS if and only if there exists a strong Dulac function for* $\mathbf{F}$ *defined in* $\mathbb{R}^2$.

Notice that the above statement unifies both sections of the paper. When we consider Condition (B) it is similar in spirit to two earlier theorems given in ([17], [18]) and collected here as Theorem 7. The above result improves a previous result of the first author; see [5].

2. **Existence of a Dulac function.** This section is devoted to proving Theorems 1, 2 and 3 and gives several examples to illustrate the sharpness of their hypotheses.

Proof of Theorem 1. Without loss of generality we assume that $\mathbf{p} = \mathbf{0}$. Consider a Liapunov function $f : N \to \mathbb{R}$, which always exists, see [20, 21], where N is the basin of attraction of the origin. We wish to construct a function $g : \mathbb{R}^+ \to \mathbb{R}^+$ such that $B := e^{g(f)}$ is a Dulac function. First note that

$$\begin{aligned} \operatorname{div}(e^{g(f)}\mathbf{F}) &= e^{g(f)}g'(f)\dot{f} + e^{g(f)}\operatorname{div}\mathbf{F} \\ &= e^{g(f)}[g'(f)\dot{f} + \operatorname{div}\mathbf{F}]. \end{aligned}$$

Since for all $\mathbf{x} \in N \setminus \{0\}$, $g(f(\mathbf{x})) > 0$, it implies that $B(x) > 1$ and hence it is enough to consider the inequality $g'(f)\dot{f} + \operatorname{div}\mathbf{F} < -k$, or equivalently $g'(f) > -(k + \operatorname{div}\mathbf{F})/\dot{f}$. Since $\operatorname{div}\mathbf{F}(0) < 0$ and $\dot{f} < 0$, we see that as $\mathbf{x}$ tends to $\mathbf{0}$, the above quotient tends to $\pm\infty$ depending on the sign of the numerator. We choose $k > 0$ such that $k < -\operatorname{div}\mathbf{F}(0)$ which assures that the limit is $-\infty$. Define ϕ as

$$\phi(h) = \max_{\mathbf{x}\in f^{-1}(h)} \left\{ \frac{-k - \operatorname{div}\mathbf{F}}{\dot{f}} \right\}.$$

Since ϕ is well-defined (recall that the level sets of f are compact) in $(0, \infty)$, continuous and $\phi(h)$ tends to $-\infty$ as h tends to zero, it is also bounded above on bounded intervals.

Now choose g analytic with $g'(h) > \max\{\phi(h), 0\}$ and let

$$g(h) = \int_0^h g'(\xi)d\xi > 0$$

for $h > 0$. It is clear that $B(x) = e^{g(f(x))}$ is a Dulac function with the property $\operatorname{div}(B\mathbf{F})(\mathbf{x}) \le -k$ for all $\mathbf{x} \in N$. Furthermore, it is clear that if f is analytic, so is $B = e^{g(f)}$. □

Now we consider the case $\operatorname{div}\mathbf{F}(\mathbf{p}) = 0$. First we have the following observation.

Lemma 1. *If* $\mathbf{F} : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ *admits LAS at* p*, then* $\operatorname{div}\mathbf{F}(\mathbf{p}) \le 0$. *Furthermore, either there exists a neighbourhood* V *of* $\mathbf{p}$ *such that* $\operatorname{div}\mathbf{F}(x) \le 0$ *for all* $\mathbf{x} \in V$ *or in any neighbourhood of* $\mathbf{p}$ *the divergence changes sign.*

Proof. If $\operatorname{div}\mathbf{F}(\mathbf{p}) > 0$, then $D\mathbf{F}(\mathbf{p})$ must have some eigenvalue with positive real part. Associated to this eigenvalue the differential equation has an unstable manifold. This contradicts the fact that $\mathbf{F}$ has LAS at p. If $\operatorname{div}\mathbf{F}(\mathbf{p}) < 0$, then clearly $\mathbf{p}$ has a neighbourhood in which the divergence is negative, so we assume that $\operatorname{div}\mathbf{F}(\mathbf{p}) = \mathbf{0}$. Suppose there exists a neighbourhood V of $\mathbf{p}$ such that $\operatorname{div}\mathbf{F}(\mathbf{x}) \ge 0$ for all $\mathbf{x} \in V$. Without loss of generality, we take V to be bounded and let f be a Liapunov function. Set

$$\delta = \min_{x\in\partial V} f(\mathbf{x})$$

and let $W = f^{-1}(\delta/2)$. Clearly W is positively invariant and so, for any $T > 0$, $\varphi(T, W)$ is strictly contained in W. Furthermore, $W \setminus \varphi(T, W)$ has positive measure. On the other hand we obtain

$$\int_{\varphi(T,W)} d\mathbf{x} = \int_W e^{\operatorname{div} F(\varphi(T,\mathbf{x}))} d\mathbf{x} \ge \int_W d\mathbf{x},$$

which gives a contradiction.

□

The next result is a first step towards proving Theorem 2.

Proposition 1. *Let $\mathbf{F}:\mathbb{R}^n \to \mathbb{R}^n$ be a C^∞ vector field and suppose that the system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ has LAS at $\mathbf{0}$ with $\operatorname{div}\mathbf{F}(\mathbf{0}) = 0$ and $\operatorname{div}\mathbf{F} \leq 0$ in a punctured neighbourhood of $\mathbf{0}$. Let f be a $\mathcal{C}^\infty$ Liapunov function defined in its basin of attraction N. Then for any $k>0$, there exists a $\mathcal{C}^\infty$ Dulac function $B: N \to \mathbb{R}$ such that $\operatorname{div}(B\mathbf{F}) < 0$ in $V \setminus \{\mathbf{0}\}$ and $\operatorname{div}(B\mathbf{F}) \leq -k$ in $N \setminus V$, where $V \subset N$ is a bounded neighbourhood of $\mathbf{0}$. Moreover, B is analytic if f is analytic.*

Proof. We consider $B = e^{g(f)}$ as before, where g must satisfy

$$g'(f)\dot{f} + \operatorname{div}\mathbf{F} < 0$$

near the origin. Let $\delta > 0$ be small enough such that $\operatorname{div}(\mathbf{F}(\mathbf{x})) \leq 0$ for all $\mathbf{x}$ with $f(\mathbf{x}) \leq \delta$ and set $V = f^{-1}([0,\delta])$. Clearly if we choose g such that $g'(h) > 0$ for all h, we obtain $\operatorname{div}(B\mathbf{F})(\mathbf{x}) < 0$ for all $\mathbf{x} \in V \setminus \{\mathbf{0}\}$. For a given $k > 0$, define

$$\phi(h) = \max_{\mathbf{x}\in f^{-1}(h)} \left\{ \frac{-k - \operatorname{div}\mathbf{F}}{\dot{f}} \right\}.$$

Let $g(h)$ be an analytic function such that $g(0) > 0$ and

$$g'(h) > \begin{cases} 0 & \text{if } h \in [0,\delta], \\ \max(\phi(h), 0) & \text{otherwise.} \end{cases}$$

Then clearly, $B(\mathbf{x}) = e^{g(f(\mathbf{x}))}$ satisfies the conditions stated. □

The next example gives a system having LAS yet not satisfying the hypotheses of the Proposition 1.

Example 1. *The system*

$$\begin{aligned} \dot{x} &= -x^3 + xy^2 + y^3, \\ \dot{y} &= -x^2y + y^3. \end{aligned}$$

has LAS at $\mathbf{0}$ and its divergence changes sign in any neighbourhood of $\mathbf{0}$.

Proof. By using the blow-up technique — see for instance [1, Chapter IX] — it is not difficult to see that the above system has an attracting node at the origin. Since the system is homogeneous, the behaviour of the orbits near the origin determines the global phase portrait, hence the origin admits GAS. On the other hand, the divergence of $\mathbf{F}(x,y) = (-x^3 + xy^2 + y^3, -x^2y + y^3)$ is

$$\operatorname{div}\mathbf{F}(x,y) = -3x^2 + y^2 - x^2 + 3y^2 = 4(y-x)(y+x),$$

which clearly changes sign in any neighbourhood of $\mathbf{0}$. □

In the next proposition we consider the situation described in Example 1: LAS with divergence changing sign near the origin, which is not covered by Proposition 1. To prove this, we need a result due to Lojasiewicz which will be stated in Lemma 2; see [19] and [4]. We also give an example (see Example 3) which shows that in general we can not hope to have analytic Dulac functions, and hence that the results of Proposition 2 concerning the regularity of the Dulac functions are sharp.

Lemma 2. *Let g be an analytic function and f a $\mathcal{C}^\infty$ function, both defined in an open set $U \subset \mathbb{R}^n$, and taking real values. Assume that f is zero on the zeroes of g. Then for all compact $K \subset U$, there exists $c \in \mathbb{R}$ and $m \in \mathbb{R}$ such that*

$$|g(\mathbf{x})| \geq c|f(\mathbf{x})|^m$$

for all $\mathbf{x} \in K$.

Proposition 2. *Let* $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *be an analytic vector field and suppose that the system* $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ *has LAS at* $\mathbf{0}$ *with* $\operatorname{div}\mathbf{F}(\mathbf{0}) = 0$. *Let* f *be a Liapunov function defined on its basin of attraction* N, *and suppose that* f *is analytic. Then there exists a* $\mathcal{C}^\infty$ *strong Dulac function defined on* N.

Proof. Let f be the analytic Liapunov function and consider $B_1(\mathbf{x}) = e^{-1/f^k(\mathbf{x})}$, where $k \in \mathbb{R}$. Then

$$\operatorname{div}(B_1\mathbf{F}) = \dot{B}_1 + B_1 \operatorname{div}\mathbf{F} = B_1\left[\frac{k\dot{f}}{f^{k+1}} + \operatorname{div}\mathbf{F}\right]. \tag{2}$$

Let $g(\mathbf{x}) = \dot{f}(\mathbf{x})$ and apply Lemma 2 to $f(\mathbf{x})$ and $g(\mathbf{x})$. Consider any compact $K \subset N$ and $c \in \mathbb{R}$, $m \in \mathbb{R}$ such that

$$|\dot{f}(\mathbf{x})| \geq c|f(\mathbf{x})|^m,$$

for all $\mathbf{x} \in K$. If $k+1 = m$, then

$$\frac{k\dot{f}(\mathbf{x})}{f^{k+1}} = -\frac{|k\dot{f}(\mathbf{x})|}{f^m(\mathbf{x})} \leq -kc,$$

for all $\mathbf{x} \in K$. Taking (2) into account and using $\operatorname{div}(\mathbf{F}(\mathbf{0})) = 0$ we see that $\operatorname{div}(B_1\mathbf{F}) < 0$ when $|\mathbf{x}|$ is sufficiently small. Note that the vector field $B_1\mathbf{F}$ is of class $\mathcal{C}^\infty$ and that we have lost the analyticity at zero, but $B_1\mathbf{F}$ has the property that $\operatorname{div}(B_1\mathbf{F})(\mathbf{0}) = 0$ and $\operatorname{div}(B_1\mathbf{F}) < 0$ when $|\mathbf{x}|$ is sufficiently small. We may thus apply Proposition 1 and obtain the existence of a strong Dulac function B_2 such that $\operatorname{div}(B_2B_1\mathbf{F}) < 0$. The function $B = B_2B_1$ is defined in N, is of class $\mathcal{C}^\infty$, and is a strong Dulac function for $\mathbf{F}$. □

Example 2. *The system*

$$\begin{aligned} \dot{x} &= -2x^3 - 2y^4 - 4xy^4, \\ \dot{y} &= xy - y^5. \end{aligned}$$

has GAS at $\mathbf{0}$, *admits an analytic Liapunov function, but it does not have any analytic Dulac function.*

Proof. The function $f(x,y) = x^2 + y^4$ satisfies $f(x,y) > 0$ for all $(x,y) \neq (0,0)$ and $\dot{f} = -4(x^2+y^4)^2 < 0$ for all $(x,y) \neq (0,0)$. Hence it is a global Liapunov function and the above system has GAS at $\mathbf{0}$. On the other hand, assume there exists an analytic Dulac function $B(\mathbf{x}) = B_k(\mathbf{x}) + B_{k+1}(\mathbf{x}) + \cdots$ where each $B_i(\mathbf{x})$ is homogeneous of degree i and k is even, and $B_k \not\equiv 0$. Setting $W(\mathbf{x}) = B_k(\mathbf{x})$, we have that

$$\begin{aligned} \operatorname{div}(B\mathbf{F}) &= \dot{B} + B\operatorname{div}\mathbf{F} \\ &= W_x\dot{x} + W_y\dot{y} + W(x - 6x^2 - 9y^4) + HOT \\ &= xyW_y + xW + HOT. \end{aligned}$$

Since $xyW_y + xW$ has odd degree $k+1$, the sign condition near the origin requires $x[yW_y + W] \equiv 0$. Solving implies $W = \phi(x)/y$ for some function ϕ. For W to be a polynomial, this requires $W \equiv 0$, a contradiction.

□

Proof of Theorem 2. It follows from Propositions 1 and 2. □

The next examples show that the regularity of the Liapunov function — class $\mathcal{C}^\infty$ for analytic vector fields (see [20]) — cannot be improved.

Example 3. *The system*

$$\begin{aligned}\dot{x} &= y(-x^3 - y^3) - x^5,\\ \dot{y} &= y(x^3 - x^2 y),\end{aligned}$$

has LAS at **0** *but does not admit an analytic Liapunov function.*

Proof. By using the blow-up technique — see for instance [1, Chapter IX] — it is easy to see that the above system has an attracting node at the origin. Now suppose that $f(\mathbf{x})$ is an analytic Liapunov function. Since $f(\mathbf{x}) \geq 0$, we can write $f(\mathbf{x}) = f_k(\mathbf{x}) + f_{k+1}(\mathbf{x}) + HOT$, where f_i is homogeneous of degree i and k must be even, and $f_k \not\equiv 0$. Setting $f_k(\mathbf{x}) = V(\mathbf{x})$ and noting that $f_x\dot{x} + f_y\dot{y} < 0$ in a punctured neighbourhood of **0**, the fact that k is even implies

$$V_x y(-x^3 - y^3) + V_y y(x^3 - x^2 y) \equiv 0.$$

By divisibility arguments there exists a homogeneous polynomial $\mu(x, y)$ such that $V_x(x, y) = \mu(x, y)(x^3 - x^2 y)$ and $V_y(x, y) = \mu(x, y)(x^3 + y^3)$. Then the system

$$\begin{aligned}\dot{x} &= -\mu(x, y)(x^3 + y^3),\\ \dot{y} &= \mu(x, y)(x^3 - x^2 y),\end{aligned} \tag{3}$$

is a Hamiltonian system. It can be seen that the associated system

$$\begin{aligned}\dot{x} &= -x^3 - y^3,\\ \dot{y} &= x^3 - x^2 y,\end{aligned}$$

has an attracting focus at **0**. Hence the phase portrait of (3) is also of focus type, eventually cut by lines of critical points given by $\mu(x, y) = 0$. This phase portrait is not compatible with the Hamiltonian character of (3), yielding the desired contradiction. □

The claim of Proposition 2 may remain true even if the Liapunov function is not analytic, as the following example shows. To construct it, we are inspired by a planar system with a center and without an analytic first integral; see [16, p. 122] and [7]. Note that it also gives an example of GAS that does not admit any analytic Liapunov function.

Example 4. *The system*

$$\begin{aligned}\dot{x} &= -y[2x^2 + y^2 + (x^2 + y^2)^2] - x^7,\\ \dot{y} &= x[2x^2 + y^2 + 2(x^2 + y^2)^2] - y^7,\end{aligned}$$

has GAS at **0**, *its divergence changes sign in any neighbourhood of the origin and it has no analytic Liapunov function. However it has a* C^∞ *Dulac function.*

Proof. The function $f(x, y) = (2x^2 + y^2)\exp\frac{-1}{x^2+y^2}$ satisfies $\dot{f}(x, y) < 0$ for all $(x, y) \neq (0, 0)$ which assures that the above system has GAS at **0**. We notice that this function is in fact the non-analytic first integral found in [16][1] for the above system when we remove the terms of degree seven. On the other hand, the Taylor expansion of the divergence begins with the term $-2xy$ and so it changes sign in any neighbourhood of **0**. To see that this system has no analytic Liapunov function, we argue by contradiction. Let L be an analytic Liapunov function of the form

$$L = L_{2k} + L_{2k+n} + HOT,$$

[1]Note that there is a typo in [16]; their exponential part has a “+1” instead of a “-1”.

where $L_{2k} \not\equiv 0$ is a polynomial. First we show that $L_{2k} = A(x^2+y^2)^k$ for some $A > 0$. This implies

$$\begin{array}{rl} \dot{L}(x,y) & = \frac{\partial L_{2k}(x,y)}{\partial x}(-y(2x^2+y^2)) + \frac{\partial L_{2k}(x,y)}{\partial y}(x(2x^2+y^2)) + HOT \\ & = (2x^2+y^2)(-y\frac{\partial L_{2k}(x,y)}{\partial x} + x\frac{\partial L_{2k}(x,y)}{\partial y}) + HOT \end{array},$$

and thus we require

$$-y\frac{\partial L_{2k}(x,y)}{\partial x} + x\frac{\partial L_{2k}(x,y)}{\partial y} \leq 0$$

for all $(x,y) \neq (0,0)$. Changing to polar coordinates gives

$$L_{2k}(r\cos\theta, r\sin\theta) = r^{2k}g(\theta),$$

where $g(\theta)$ is a trigonometric polynomial of degree $2k$. Note also that

$$-y\frac{\partial L_{2k}(r\cos\theta, r\sin\theta)}{\partial x} + x\frac{\partial L_{2k}(r\cos\theta, r\sin\theta)}{\partial y} = r^{2k}g'(\theta),$$

thus we require $g'(\theta) \leq 0$. Since is $g(\theta)$ periodic, this implies that g is constant, and since L must be positive we obtain $L_{2k}(x,y) = Ar^{2k}$ for some $A > 0$. Without loss of generality, we assume that $A = 1$, implying $g(\theta) \equiv 1$.

Now we claim that $n = 2$, for if $n = 1$, this would imply

$$\dot{L}(x,y) = \left(-y\frac{\partial L_{2k+1}(x,y)}{\partial x} + x\frac{\partial L_{2k+1}(x,y)}{\partial y}\right)(2x^2+y^2) + HOT.$$

Arguing as above, we obtain that L_{2k+1} does not depend on θ which contradicts that $2k+1$ is odd.

Now suppose $n > 2$. This implies

$$\dot{L}(x,y) = 2k(x^2+y^2)^{k+1}xy + HOT,$$

which changes sign, hence $n = 2$. As above, put $L_{2k+2}(r\cos\theta, r\sin\theta) = r^{2k+2}h(\theta)$, where $h(\theta)$ is a trigonometric polynomial of degree $2k+2$. Imposing that $\dot{L}$ must be negative, we obtain the following inequality:

$$\ell(\theta) := h'(\theta) + \frac{2k\cos\theta\sin\theta}{1+\cos^2\theta} \leq 0.$$

Integrating $\ell(\theta)$ between 0 and 2π yields

$$\int_0^{2\pi} \ell(\psi)\,d\psi = h(2\pi) - h(0) - 2k\log\left(\frac{1+\cos^2(2\pi)}{1+\cos^2 0}\right) = 0.$$

Since ℓ is a continuous function, we have $\ell = 0$, hence

$$\ell(\theta) = h'(\theta) + \frac{2k\cos\theta\sin\theta}{1+\cos^2\theta} \equiv 0$$

for all θ. This forces $h(\theta) = \log(1+\cos^2\theta) + K$ for some constant K, which contradicts that h is a trigonometric polynomial. This proves that our system does not admit an analytic Liapunov function.

To see that the system has a Dulac function, multiply the system by $B_1(\mathbf{x}) = \exp\frac{-1}{f(\mathbf{x})}$. The divergence of $B_1\mathbf{F}$ equals $B_1[\frac{\dot{f}}{f^2} + \operatorname{div}\mathbf{F}]$ and

$$\frac{\dot{f}}{f^2} = \frac{-2\exp\frac{-1}{x^2+y^2}[(2x^8+y^8)(x^2+y^2)^2 + (x^8+y^8)(2x^2+y^2)]}{\exp\frac{-2}{x^2+y^2}(2x^2+y^2)^2(x^2+y^2)^2} \to -\infty$$

as $(x,y) \to (0,0)$. This implies we can apply to this system the arguments used in the proof of Proposition 1 and conclude that it has a C^∞ Dulac function. $\square$

Proof of Theorem 3. As usual, we assume $\mathbf{p} = \mathbf{0}$. Let f be a C^∞ Liapunov function defined in N. Let $T : N \setminus \{\mathbf{0}\} \longrightarrow \mathbb{R}$ be defined implicitly by $f(\varphi(T(\mathbf{x}), \mathbf{x})) = 1$. By the implicit function theorem, T is C^∞ on $N \setminus \{\mathbf{0}\}$. Clearly

$$\lim_{\mathbf{x}\to\mathbf{0}} T(\mathbf{x}) = -\infty.$$

Note also that

$$\begin{aligned}\dot T(\mathbf{x}) = &\ \left.\frac{dT(\varphi(t,\mathbf{x}))}{dt}\right|_{t=0} = \left.\left(\lim_{h\to 0}\frac{T(\varphi(t+h,\mathbf{x}))-T(\varphi(t,\mathbf{x}))}{h}\right)\right|_{t=0}\\ = &\ \left.\left(\lim_{h\to 0}\frac{T(\varphi(t,\mathbf{x}))-h-T(\varphi(t,\mathbf{x}))}{h}\right)\right|_{t=0} = -1\end{aligned}$$

for all $\mathbf{x} \in N \setminus \{\mathbf{0}\}$. Set

$$B_1(\mathbf{x}) = \begin{cases} e^{T(\mathbf{x})} & \text{if } \mathbf{x} \neq \mathbf{0},\\ 0 & \text{if } \mathbf{x} = \mathbf{0}.\end{cases}$$

Clearly $B_1\mathbf{F}$ is continuous on N and C^∞ on $N \setminus \{\mathbf{0}\}$. Since for all $\mathbf{x} \in N \setminus \{\mathbf{0}\}$ we have

$$\operatorname{div}(B_1\mathbf{F})(\mathbf{x}) = B_1(\mathbf{x})(-1 + \operatorname{div}(\mathbf{F})(x))$$

and $\operatorname{div}(\mathbf{F})(\mathbf{0}) \leq 0$, we obtain the existence of a punctured neighbourhood of $\mathbf{0}$ where the divergence of $B_1\mathbf{F}$ remains negative. Now adapting the proof of Proposition 2.1 yields the desired result. □

Remark 2. Consider an analytic vector field $\mathbf{F}$, and a $\mathcal{C}^\infty$ function f. If we were able to prove a result similar to Lemma 2, but just for functions g of the form $g(\mathbf{x}) := \dot f(\mathbf{x}) = \nabla f(x) \cdot \mathbf{F}(x)$, then the proof of Proposition 2 could be adapted to the case when the Liapunov function is only $\mathcal{C}^\infty$, hence the continuity at the critical point of the Dulac function given in Theorem 3 could be improved to yield a $\mathcal{C}^\infty$ Dulac function. Note that this fits the setting of Example 4.

3. **Sufficient Conditions for GAS in the Plane.** Determining whether a finite-dimensional system $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ admits GAS has always been a challenging problem. The most common approach is to use the approach of Liapunov. However, constructing the required Liapunov function, especially for non-physical systems, is difficult. Various other sufficiency criteria have been established for special classes of systems, based on nullcine analysis, the elimination of periodic orbits via the construction of a Dulac function, and the Poincaré-Bendixson Theorem.

In 1960, Markus and Yamabe[15] posed a conjecture which would imply GAS under conditions which are easy to check. The conditions are based on $D\mathbf{F}$, the Jacobian matrix of $\mathbf{F}$.

Conjecture 1. (Markus-Yamabe)
Let the C^1 map $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ and a point $\mathbf{p} \in \mathbb{R}^n$ satisfy the following conditions:
1. $\mathbf{F}(\mathbf{p}) = \mathbf{0}$
2. *Real*$(\lambda) < 0$ *for all eigenvalues* λ *of* $D\mathbf{F}(\mathbf{x})$ *for all* $\mathbf{x} \in \mathbb{R}^n$.

Then $\mathbf{p}$ *is GAS.*

Condition 2 clearly implies local asymptotic stability (LAS). Finally settled in the 1990s, the Markus-Yamabe Conjecture is true if $n = 2$ ([10], [11], [12]) and false if $n \geq 3$ ([2], [8]). In two dimensions, condition 2 is equivalent to having $\operatorname{Trace} D\mathbf{F}(\mathbf{x}) = \operatorname{div}(\mathbf{F}(\mathbf{x})) < 0$ and $\operatorname{Det} D\mathbf{F}(\mathbf{x}) > 0$ for all $\mathbf{x} \in \mathbb{R}^2$. As mentioned earlier, the trace condition may be geometrically interpreted as the shrinking of finite areas in forward time. The determinant condition does not admit any obvious geometrical interpretation.

Results have been obtained where this determinant condition has been essentially dropped, while adding a condition in a neighborhood of infinity. Combining two earlier theorems ([17], [18]) yields the following general result.

Theorem 7. *Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field having a unique critical point $\mathbf{p}$ which is locally asymptotically stable. Furthermore, assume the following conditions:*

C_1**:** *: div ($\mathbf{F}$)≤ 0 for all $\mathbf{x} \in \mathbb{R}^2$.*

C_2**:** *: There exists an $A > 0$ such that one of the following conditions hold for all $|\mathbf{x}| > A$:*

- I_1**:** *: $|\mathbf{F}(\mathbf{x})| > B$ for some $B > 0$.*
- I_2**:** *: The symmetric part of $D\mathbf{F}(\mathbf{x})$ is non-negative definite.*

Then $\mathbf{p}$ is GAS.

In Chamberland *et al.*[6], the authors also consider C^1 vector fields $\mathbf{F}$ having a unique critical point $\mathbf{p}$ which is LAS and satisfies the condition

C_3**:** : div $\mathbf{F} < \mathbf{0}$ for all $\mathbf{x} \in \mathbb{R}^2$,

and ask whether such conditions imply GAS. They prove that one indeed obtains GAS for quadratic systems and Liénard systems, but found the following analytic counter-example:

Example 5.

$$\begin{aligned} \dot{x} &= \frac{-x(x+1)}{(1+y^2)^{3/2}}, \\ \dot{y} &= 4x + \frac{(2x-1)y}{\sqrt{1+y^2}}. \end{aligned}$$

Example 5 is part of a more general class of non-polynomial systems. To narrow the gap between such examples which do not admit GAS and the quadratic systems which do, we offer the following polynomial counter-example.

Example 6. *For systems of the form*

$$\begin{aligned} \dot{x} &= -x + 3ax^2y^2 + 5x^3y^4, \\ \dot{y} &= -ky - 2axy^3 - 3x^2y^5, \end{aligned} \tag{4}$$

where $a = -(5k+4)/\sqrt{3+3k}$ and $k > 0$, the origin is the unique equilibrium point, it is LAS, the divergence is a negative constant and there is no GAS.

Proof. First show that the origin is the unique equilibrium point, that is, $(x, y) = (0, 0)$ is the unique point satisfying

$$-x + 3ax^2y^2 + 5x^3y^4 = 0, \ -ky - 2axy^3 - 3x^2y^5 = 0.$$

Note that $x = 0$ if and only if $y = 0$. If neither x nor y equal zero, then

$$-1 + 3axy^2 + 5x^2y^4 = 0, \ -k - 2axy^2 - 3x^2y^4 = 0.$$

Linear combinations of these equations force

$$xy^2 = \frac{-3-5k}{a}, \ x^2y^4 = 3k + 2,$$

thus

$$3k + 2 = \left(\frac{-3-5k}{a}\right)^2.$$

Using the definition of a yields $5k^2 + 11k + 5 = 0$, contradicting $k > 0$, therefore the origin is the unique equilibrium point of the system. The Jacobian matrix is

$$\begin{pmatrix} -1 + 6axy^2 + 15x^2y^4 & 6ax^2y + 20x^3y^3 \\ -2ay^3 - 6xy^5 & -k - 6axy^2 - 15x^2y^4 \end{pmatrix}.$$

The trace of this matrix equals $-1 - k < 0$ for all (x, y). The determinant equals k at $(0, 0)$, hence the origin is LAS.

The structure of the system allows us to find a solution. The curve defined implicitly by $xy^2 = \sqrt{3 + 3k}$ is a solution of the system. Since it is bounded away from the origin, the system does not admit GAS. □

Remark 3. The aim of this remark is to give the key steps used to obtain the example given in the previous theorem. First, note that a wide family of systems with constant negative divergence is given by

$$\begin{aligned} \dot{x} &= -x + \frac{\partial H(x,y)}{\partial y}, \\ \dot{y} &= -ky - \frac{\partial H(x,y)}{\partial x}, \qquad \text{with k} < 0. \end{aligned}$$

The second step is to find an H in such a way that it is easy to prove that the origin is the only critical point and furthermore, that it is not a global attractor. This second property is easy to check when we have a quasi-homogeneous vector field. In fact, this was how the polynomial counterexample to the Markus-Yamabe Conjecture was found; see [8] and [9]. To this end, we consider $H(x, y) = ax^2y^3 + x^3y^5$. Notice that the vector field is quasi-homogeneous because $H(\lambda^2 x, \lambda^{-1} y) = \lambda H(x, y)$. To fix the value of a, we need only impose that the origin is its only critical point and that it admits a solution of the form $(x(t), y(t)) = (x_0 \exp(2t), y_0 \exp(-t))$.

In an attempt to obtain sufficient conditions for GAS which mirror the results obtained in Theorems 1, 2 and 3, we obtain Theorem 6 which is similar in spirit to Theorem 7, but with a condition in a neighbourhood of infinity which uses the curvature of orbits. Our result improves a previous result of the first author; see [5]. In order to prove Theorem 6 we get some other results about GAS. All of these results are stated in the introduction of the paper as Theorem 4 and we prove them in the sequel.

Proof of Theorem 4. First we prove the "only if" part. The existence of an SAI implies the existence of an orbit that escapes to infinity in positive time, which implies that $\mathbf{p}$ is not globally asymptotically stable.

Now we prove the "if" part. Suppose $\mathbf{p}$ is not GAS. Let N be its basin of attraction and $\mathbf{q}$ belong to the boundary of N. Since from the hypotheses there are no periodic orbits, the Poincaré-Bendixon Theorem implies $\gamma_+(\mathbf{q})$ and $\gamma_-(\mathbf{q})$ escape to infinity. Let Γ be a Jordan curve contained in N containing $\mathbf{p}$ in its interior and without contact with the solutions of the vector field. Clearly the interior of Γ is positive invariant. Let Σ be a transversal section to $\gamma(\mathbf{q})$ at $\mathbf{q}$. Note that for each point $\mathbf{x} \in \mathbf{\Sigma} \cap \mathbf{N}$ the positive orbit of $\mathbf{x}$ must intersect Γ in a unique point $\Pi(\mathbf{x})$. Since Γ has no contact with the vector field, the map

$$\Pi : \Sigma \cap N \longrightarrow \Gamma$$

must be continuous and locally injective. Hence its elevation $\tilde{\Pi} : \Sigma \cap N \longrightarrow \mathbb{R}$ must be monotone. Note also that if $\Pi(\mathbf{x}) = \Pi(\mathbf{y})$ then $\gamma(\mathbf{x}) = \gamma(\mathbf{y})$. Then there are two possibilities:

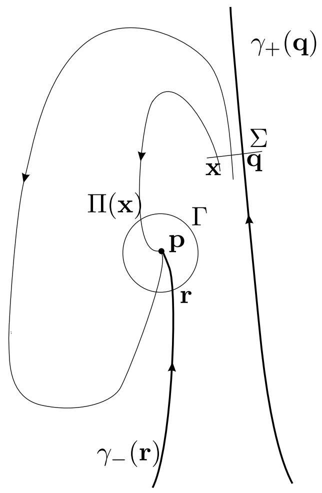

FIGURE 2. Construction of a SAI when $\Pi(\mathbf{x})$ has limit.

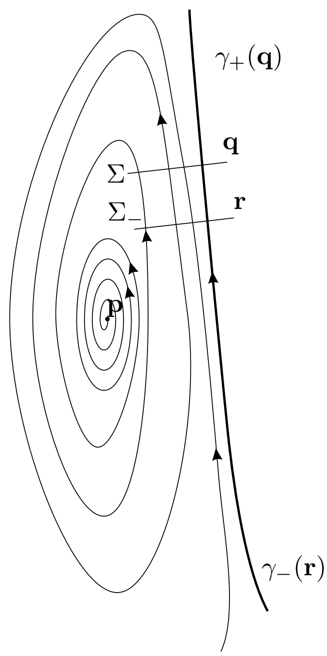

FIGURE 3. Construction of a SAI when $\Pi(\mathbf{x})$ has no limit.

Case I. The limit $\lim_{\mathbf{x}\to\mathbf{q}} \Pi(\mathbf{x})$ exists. Setting $\mathbf{r} = \lim_{\mathbf{x}\to\mathbf{q}} \Pi(\mathbf{x})$, the Poincaré-Bendixon Theorem implies the α-limit of $\mathbf{r}$ is empty, therefore $(\gamma_+(\mathbf{q}), \gamma_-(\mathbf{r}))$ is clearly (see Figure 2) an SAI.

Case II. The limit $\lim_{\mathbf{x}\to\mathbf{q}} \Pi(\mathbf{x})$ does not exist. This implies (see Figure 3) that for any $\mathbf{x} \in \Sigma \cap N$ the negative orbit of $\mathbf{x}$ cuts Σ infinitely many times. This implies its α-limit set contains $\gamma(\mathbf{q})$. Considering $\mathbf{r} = \varphi(t, \mathbf{q})$ with $t < 0$, we obtain $(\gamma_+(\mathbf{q}), \gamma_-(\mathbf{r}))$ is an SAI.

This completes the proof of the theorem. □

From Theorem 4 we obtain an easy test to check whether a planar vector field having a unique critical point which is LAS and without periodic orbits is indeed GAS: it suffices to prevent the existence of SAI.

For proving Theorem 5 the following construction will be useful:

Definition 1. Let $\gamma_+(\mathbf{q}), \gamma_-(\mathbf{r})$ be an SAI for $\mathbf{F}$ and Σ_+ and Σ_- its two transversal sections. For fixed $\mathbf{x}, \mathbf{y} \in \Sigma_+$ let $L_\mathbf{x}$ and $L_\mathbf{y}$ denote the positive orbits of $\mathbf{x}$ and $\mathbf{y}$ ending at $\pi(\mathbf{x})$ and $\pi(\mathbf{y})$, respectively. We call $\alpha^+_{\mathbf{x},\mathbf{y}}$ the curve in Σ^+ joining $\mathbf{x}$ and $\mathbf{y}$ and $\alpha^-_{\mathbf{x},\mathbf{y}}$ the curve in Σ^- joining $\pi(\mathbf{x})$ and $\pi(\mathbf{y})$. The four curves $L_\mathbf{y}, \alpha^+_{\mathbf{x},\mathbf{y}}, L_\mathbf{x}, \alpha^-_{\mathbf{x},\mathbf{y}}$ delimit a simply connected compact region in $\mathbb{R}^2$ that we call a **canonical region associated to the SAI**. We denote it by $D_{\mathbf{x},\mathbf{y}}$, and its counterclockwise-oriented boundary by $\partial D_{\mathbf{x},\mathbf{y}}$. If $\mathbf{s}$ is the endpoint of Σ_+ distinct from $\mathbf{q}$, we define the **interior of the SAI** by

$$D_{\mathbf{q},\mathbf{r}} := \cup_{\mathbf{x}\in\Sigma_+} D_{\mathbf{s},\mathbf{x}}.$$

Note that the interior of an SAI is an unbounded region that contains Σ_+, Σ_-, $\gamma_+(\mathbf{q}), \gamma_-(\mathbf{r})$ and $L_\mathbf{s}$ in its boundary

We say that an SAI is a **saddle at infinity with finite area** (SAIF) if its interior has finite area. Letting $A_{\mathbf{x},\mathbf{y}} = \text{area}\,(D_{\mathbf{x},\mathbf{y}})$, the set $\{A_{\mathbf{x},\mathbf{y}} : \mathbf{x}, \mathbf{y} \in \Sigma_+\}$ is bounded and

$$\text{area}\,(D_{\mathbf{q},\mathbf{r}}) = \lim_{\mathbf{x}\to\mathbf{q}} A_{\mathbf{s},\mathbf{q}}.$$

We call this number the area of the SAIF. Note that this number depends on the selection of the transversal sections while the notion of SAIF is independent of this selection.

Proposition 3. *Let $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 vector field that has a strong Dulac function B and assume that it has an SAI. Then this SAI has to be a saddle at infinity with finite area.*

Proof. Let $K > 0$ and $R > 0$ be such that for all $\mathbf{x}$ with $\|\mathbf{x}\| \geq R$,

$$\text{div}\,(B\mathbf{F})(\mathbf{x}) < -K.$$

Assume that $\gamma_+(\mathbf{q}), \gamma_-(\mathbf{r})$ is an SAI for $\mathbf{F}$ and let Σ_+ and Σ_- be the transversal sections at $\mathbf{q}$ and $\mathbf{r}$ associated to this SAI. Let $\mathbf{s}$ be the endpoint of Σ_+ distinct from $\mathbf{q}$.

For $\mathbf{x} \in \boldsymbol{\Sigma}_+$, consider $\alpha^+_{\mathbf{s},\mathbf{x}}$ and $\alpha^-_{\mathbf{s},\mathbf{x}}$ as in Definition 1. Green's Theorem yields

$$\begin{aligned}\int\int_{D_{\mathbf{s},\mathbf{x}}} \text{div}\,(B\mathbf{F}) &= \int_{L_\mathbf{s}} (B\mathbf{F})^\perp + \int_{\alpha^-_{\mathbf{s},\mathbf{x}}} (B\mathbf{F})^\perp + \int_{L_\mathbf{x}} (B\mathbf{F})^\perp + \int_{\alpha^+_{\mathbf{s},\mathbf{x}}} (B\mathbf{F})^\perp \\ &= \int_{\alpha^-_{\mathbf{s},\mathbf{x}}} (B\mathbf{F})^\perp + \int_{\alpha^+_{\mathbf{s},\mathbf{x}}} (B\mathbf{F})^\perp.\end{aligned}$$

Denote by D the closed ball with radius R centered at the origin and by $\overline{D^C}$ the closure of $\mathbb{R}^n \setminus D$. From the above equality we obtain

$$\begin{aligned}K\int\int_{D_{\mathbf{s},\mathbf{x}}\cap\overline{D^C}} 1 &\leq -\int\int_{D_{\mathbf{s},\mathbf{x}}\cap\overline{D^C}} \text{div}\,(B\mathbf{F})(\mathbf{x}) \\ &= -\int_{\alpha^-_{\mathbf{s},\mathbf{x}}} (B\mathbf{F})^\perp - \int_{\alpha^+_{\mathbf{s},\mathbf{x}}} (B\mathbf{F})^\perp + \int\int_{D_{\mathbf{s},\mathbf{x}}\cap D} \text{div}\,(B\mathbf{F})(\mathbf{x}).\end{aligned}$$

Since the right term of this inequality is bounded we obtain that the area of $D_{\mathbf{s},\mathbf{x}} \cap \overline{D^C}$ is also bounded which obviously implies that the area of $D_{\mathbf{s},\mathbf{x}}$ is bounded on $\mathbf{x}$. Thus

$$\text{area}\,(D_{\mathbf{q},\mathbf{r}}) = \lim_{\mathbf{x}\to\mathbf{q}} A_{\mathbf{s},\mathbf{x}} < \infty,$$

and the proposition follows. □

The proof of the following lemma is straightforward.

Lemma 3. *Let D be a closed disc of radius r. Assume that γ is a smooth curve contained in D such that $\gamma \cap \partial D \neq \emptyset$. Then for all $\mathbf{x} \in \gamma \cap \partial D$ the curvature of γ at $\mathbf{x}$ is no less than $1/r$.*

Finally recall that given a vector field $\mathbf{F}$ the curvature of the orbits solution of

$$\dot{x} = f(x, y),\ \dot{y} = g(x, y), \tag{5}$$

passing through an non-equilibrium point (x, y) is

$$H(x, y) = \frac{f^2 g_x + fg(g_y - f_x) - g^2 f_y}{(f^2 + g^2)^{3/2}}. \tag{6}$$

Proof of Theorem 5. First we prove A). Any planar polynomial vector field $\mathbf{F}$ can be extended to a vector field on the sphere S^2 through the so-called Poincaré compactification; see, for instance, [3] or [13]. The behaviour of this new vector field near the equator of S^2 gives information about the behaviour of $\mathbf{F}$ near infinity. The critical points on the equator are called infinite critical points. We assume that all these critical points are elementary and that $\mathbf{p}$ is not a global attractor. From Theorem 4 and Proposition 3 the vector field must have an SAIF. Since the area of the interior of the SAI is finite the two separatrices $\gamma_+(\mathbf{q})$ and $\gamma_-(\mathbf{r})$ must escape to the same critical point at infinity. This implies that there is a critical point at infinity having a hyperbolic sector for which the two separatrices are not contained in the equator of the Poincaré compactication. Such behaviour is not possible for elementary critical points, because the equator is also an invariant line through the critical point. This gives a contradiction and thus $\mathbf{p}$ is a global attractor.

Now we prove B). Assume first that the second condition holds, and let C_1 and C_2 be such that $|H(\mathbf{x})| \leq C_2$ for all $\mathbf{x}$ with $\|\mathbf{x}\| > C_1$. Suppose also that $\mathbf{p}$ is not a global attractor. Again Theorem 4 and Proposition 3 imply that the vector field $\mathbf{F}$ has an SAIF. We denote its area by A.

We now wish to show that the SAIF must contain an orbit whose curvature exceed C_2, contradicting the hypotheses and hence giving the desired GAS. Define r_1 as the maximum of C_1 and $|\mathbf{x}|$ for points $\mathbf{x}$ on the transversal sections of the SAIF. Also let

$$r = \frac{1}{1 + C_2},\ n = \left\lceil \frac{A}{\pi r^2} \right\rceil + 1,\ r_2 = r_1 + 2rn.$$

Choose $\mathbf{y} \in \Sigma_+$ such that the orbit through this point travels outside the circle $|\mathbf{x}| = r_2$. All this allows us to carve the annulus $r_1 < |\mathbf{x}| < r_2$ into n narrower annuli, namely $r_1 + 2rj < |\mathbf{x}| < r_1 + 2r(j + 1)$, $j = 0, 1, \ldots, n - 1$. Let Γ be the positive path-orbit between $\mathbf{y}$ and $\pi(\mathbf{y})$. Each of these annuli intersects Γ in at least two segments and contains a disc of radius r; see Figure 4.

Not all of these discs, which never overlap, can be in the interior of the SAIF, for if they were, this would give

$$A > n(\pi r^2) = \left(\left\lceil \frac{A}{\pi r^2} \right\rceil + 1\right)(\pi r^2) \geq A,$$

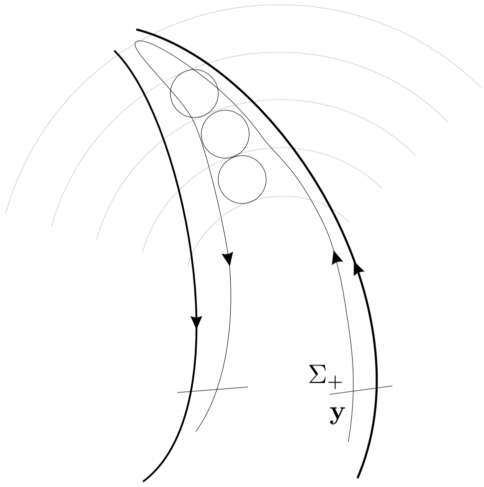

FIGURE 4. Discs in the SAIF.

a contradiction. This implies at least one of the discs cannot be moved within its annulus to lie entirely within the basin of attraction. By continuity, the disc may be positioned so that it transversely intersects a portion of Γ_2 in two places, with a (finite) arc lying exterior to the disc; see Figure 5.

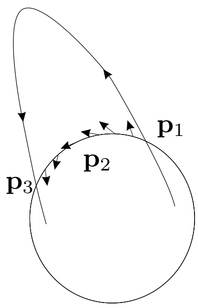

FIGURE 5. Orbits near intersections in the circle.

The orientation of the orbit crossing the circle changes at $\mathbf{p}_1$ and $\mathbf{p}_3$. Moving along the circle from $\mathbf{p}_1$ to $\mathbf{p}_3$, there must exist a first point $\mathbf{p}_2$ such that the orbit through $\mathbf{p}_2$ is tangent to the circle. The flow near $\mathbf{p}_2$ implies that the orbit through $\mathbf{p}_2$ is locally contained inside the circle. From Lemma 3 we get that the curvature of this orbit at $\mathbf{p}_2$ is no less than $1/r = 1 + C_2$. This gives the desired contradiction and hence the theorem follows. □

Proof of Remark 1. Consider the coordinates (θ, ρ) where $\rho = 1/r$ and (θ, r) are the usual polar coordinates. Let $R(r, \theta)$ and $S(r, \theta)$ be defined by

$$R(r, \theta) = \cos\theta\, Q(r\cos\theta, r\sin\theta) - \sin\theta\, P(r\cos\theta, r\sin\theta),$$
$$S(r, \theta) = \cos\theta\, P(r\cos\theta, r\sin\theta) + \sin\theta\, Q(r\cos\theta, r\sin\theta).$$

Since R and S are polynomials in r, we can write

$$R(r,\theta)=\sum_{i=1}^{n} R_i(\theta)r^i \quad \text{and} \quad S(r,\theta)=\sum_{i=1}^{n} S_i(\theta)r^i,$$

with

$$\begin{aligned} R_i(\theta) &= \cos\theta\, Q_i(\cos\theta,\sin\theta)-\sin\theta\, P_i(\cos\theta,\sin\theta),\\ S_i(\theta) &= \cos\theta\, P_i(\cos\theta,\sin\theta)+\sin\theta\, Q_i(\cos\theta,\sin\theta). \end{aligned}$$

After a reparametrization, the vector field at infinity is expressed by:

$$\begin{aligned} \dot\theta =& R_n(\theta)+R_{n-1}(\theta)\rho+\cdots+R_1(\theta)\rho^{n-1}+R_0\rho^n,\\ \dot\rho =& -S_n(\theta)\rho-S_{n-1}(\theta)\rho^2-\cdots-S_1(\theta)\rho^n-S_0\rho^{n+1}. \end{aligned}$$

We denote by $\tilde X$ the vector field with components $(\dot\theta,\dot\rho)$. When $\rho=0$, $\dot\rho=0$ and the roots of $R_n(\theta)=0$ give the directions at which the orbits of X come or reach infinity. The linear part of $\tilde X$ at $(\theta^*,0)$, with $R_n(\theta^*)=0$, is

$$\begin{pmatrix} R_n'(\theta^*) & R_{n-1}(\theta^*)\\ 0 & -S_n(\theta^*) \end{pmatrix}$$

and the eigenvalues of this matrix are $R_n'(\theta^*)$ and $-S_n(\theta^*)$. Thus the infinite critical point is elementary if and only if $(R_n'(\theta^*))^2+(S_n(\theta^*))^2\neq 0$, as we wanted to prove. □

Remark 4. (i) Notice that the system with constant negative divergence and not exhibiting GAS given in Example 6, illustrates the importance of hypotheses (A) or (B) in Theorems 5 and 6. For example, the orbits of the hyperbolic sector at infinity between the positive $y-$axis and the invariant curve $xy^2=\sqrt{3+3k}$ exemplify the high curvatures of orbits in the basin of attraction. Observe also that the critical points at infinity of our example are very degenerate.

(ii) Condition (B) is most noteworthy because the curvature of orbits is invariant if a system is scaled by a Dulac function. Suppose one had a polynomial system which admitted GAS. If one scales the system by the function $\exp(-x^2-y^2)$, condition C_2 in Theorem 7 is not met. Such a scaling, however, does not affect an application of Theorems 5 and 6.

(iii) Theorem 6 shows that under the conditions, the existence of a Dulac function is equivalent to the existence of a Liapunov function.

(iv) Note that the condition on the curvature given in Theorems 5 and 6 holds for most polynomial systems. This can be verified by considering the degrees of the numerator and denominator in H; see (6).

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