Arctangent Formulas and Pi

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Abstract. Using both geometrical and analytical approaches, new multivariable formulas connecting the arctangent function and the number π are produced.

1. INTRODUCTION. Since the discovery of Machin's formula

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right), \tag{1}$$

the arctangent function has been ubiquitous in calculations of π. While formulas like (1) have been heavily explored [**1**], we seek formulas that link π with a linear combination of arctangents of general arguments. The simplest example is the well-known equation

$$\frac{\pi}{2} = \arctan(x) + \arctan\left(\frac{1}{x}\right) \tag{2}$$

for all $x > 0$. Another example, a variant of an equation due to Euler, states

$$\frac{\pi}{2} = \arctan(x) - \arctan(x-y) + \arctan\left(\frac{x^2 - xy + 1}{y}\right)$$

for all x and when $y > 0$. The goal of this note is to develop arctangent formulas with several variables.

2. GEOMETRY OF TRIANGLES AND TETRAHEDRA. This study started serendipidously by considering the inscribed circle in a general triangle: see Figure 1. The area of the triangle can be computed in two ways. By dissecting the triangle into

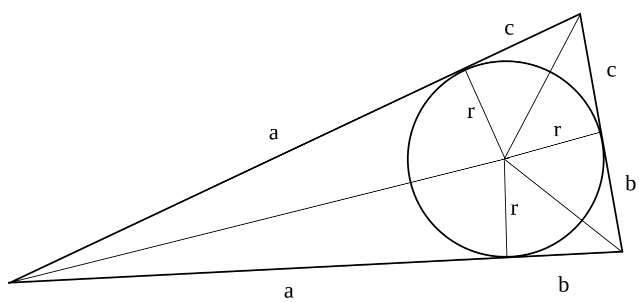

Figure 1. Inscribed circle in a triangle.

three subtriangles, we find that its total area A satisfies

$$A = \frac{1}{2}(a+c)r + \frac{1}{2}(a+b)r + \frac{1}{2}(b+c)r = (a+b+c)r,$$

where r is the radius of the inscribed circle. Alternatively, applying Heron's formula to the original triangle yields

$$A = \sqrt{abc(a+b+c)}.$$

Setting the two expressions equal produces

$$r = \sqrt{\frac{abc}{a+b+c}}.$$

Since the six angles surrounding the center of the inscribed circle sum to 2π, this produces

$$\begin{aligned}\pi = \arctan\left(a\sqrt{\frac{a+b+c}{abc}}\right) + \arctan\left(b\sqrt{\frac{a+b+c}{abc}}\right) \\ + \arctan\left(c\sqrt{\frac{a+b+c}{abc}}\right)\end{aligned} \tag{3}$$

for all $a, b, c > 0$.

To generalize this geometric approach, one could consider an $(n-1)$-sphere inscribed in a simplex in n dimensions. The volume of the simplex can be calculated with the Cayley–Menger determinant. More challenging is the generalization of the angles around the sphere's center, sometimes called "solid angles"; see [**3, 4**]. The complexity of this approach, particularly in higher dimensions, suggests an analytic approach for finding formulas similar to equation (3).

3. ARCTANGENT AND SYMMETRIC POLYNOMIALS. Some beautiful identities connect the tangent function with symmetric polynomials. Let $x_i = \tan(\theta_i)$ for $i = 1, 2, 3, \ldots$ and let $e_k(x)$ denote the kth elementary symmetric polynomial in the variables $x_1, x_2, x_3, \ldots$. The first few examples are

$$e_0(x) = 1, \quad e_1(x) = \sum_i x_i, \quad e_2(x) = \sum_{i<j} x_i x_j, \quad e_3(x) = \sum_{i<j<k} x_i x_j x_k.$$

Then a little-known formula [**2, 5**] is

$$\tan\left(\sum_i \theta_i\right) = \frac{e_1(x) - e_3(x) + e_5(x) - \cdots}{e_0(x) - e_2(x) + e_4(x) - \cdots}. \tag{4}$$

If there are only finitely many θ_i that are nonzero, then the right side of this identity is also finite. Examples are

$$\tan(\theta_1 + \theta_2) = \frac{e_1(x)}{e_0(x) - e_2(x)} = \frac{x_1 + x_2}{1 - x_1 x_2}; \tag{5}$$

$$\tan(\theta_1 + \theta_2 + \theta_3) = \frac{e_1(x) - e_3(x)}{e_0(x) - e_2(x)} = \frac{(x_1 + x_2 + x_3) - x_1 x_2 x_3}{1 - (x_1 x_2 + x_1 x_3 + x_2 x_3)}; \tag{6}$$

$$\tan(\theta_1+\theta_2+\theta_3+\theta_4) = \frac{e_1(x)-e_3(x)}{e_0(x)-e_2(x)+e_4(x)}$$
$$= \frac{(x_1+x_2+x_3+x_4)-(x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4)}{1-(x_1x_2+x_1x_3+x_1x_4+x_2x_3+x_2x_4+x_3x_4)+x_1x_2x_3x_4}. \quad (7)$$

Special choices of the angles produce formulas of the type that we seek. For example, setting $\theta_1+\theta_2=\pi/2$ in equation (5) occurs exactly when $x_1x_2=1$. If $x_1 = y_1 f(y_1,y_2)$ and $x_2 = y_2 f(y_1,y_2)$, then $f(y_1,y_2)=1/\sqrt{y_1y_2}$ for $y_i>0$ for all i. This produces

$$\frac{\pi}{2} = \arctan\left(\frac{y_1}{\sqrt{y_1y_2}}\right)+\arctan\left(\frac{y_2}{\sqrt{y_1y_2}}\right) \quad (8)$$
$$= \arctan\left(\sqrt{\frac{y_1}{y_2}}\right)+\arctan\left(\sqrt{\frac{y_2}{y_1}}\right).$$

Of course this is equivalent to equation (2).

This approach can be generalized by letting $x_i = y_i f(y_1,y_2,\ldots,y_n)$ for $i = 1,2,\ldots,n$. Setting the numerator of equation (6) equal to zero produces

$$\pi = \arctan\left(y_1\sqrt{\frac{y_1+y_2+y_3}{y_1y_2y_3}}\right)+\arctan\left(y_2\sqrt{\frac{y_1+y_2+y_3}{y_1y_2y_3}}\right)$$
$$+\arctan\left(y_3\sqrt{\frac{y_1+y_2+y_3}{y_1y_2y_3}}\right), \quad (9)$$

which is the same as equation (3), while setting the denominator of equation (6) equal to zero gives

$$\frac{\pi}{2} = \arctan\left(\frac{y_1}{\sqrt{y_1y_2+y_2y_3+y_3y_1}}\right)+\arctan\left(\frac{y_2}{\sqrt{y_1y_2+y_2y_3+y_3y_1}}\right)$$
$$+\arctan\left(\frac{y_3}{\sqrt{y_1y_2+y_2y_3+y_3y_1}}\right). \quad (10)$$

Equation (7), with its numerator equal to zero, generates

$$\pi = \arctan\left(y_1\sqrt{\frac{y_1+y_2+y_3+y_4}{y_1y_2y_3+y_1y_2y_4+y_1y_3y_4+y_2y_3y_4}}\right)$$
$$+\arctan\left(y_2\sqrt{\frac{y_1+y_2+y_3+y_4}{y_1y_2y_3+y_1y_2y_4+y_1y_3y_4+y_2y_3y_4}}\right)$$
$$+\arctan\left(y_3\sqrt{\frac{y_1+y_2+y_3+y_4}{y_1y_2y_3+y_1y_2y_4+y_1y_3y_4+y_2y_3y_4}}\right)$$
$$+\arctan\left(y_4\sqrt{\frac{y_1+y_2+y_3+y_4}{y_1y_2y_3+y_1y_2y_4+y_1y_3y_4+y_2y_3y_4}}\right). \quad (11)$$

It is natural to wonder whether there are other “simple” choices of f that produce equations of the form

$$\begin{aligned} C = \arctan\left(y_1 f(y_1, y_2, \ldots, y_n)\right) + \arctan\left(y_2 f(y_1, y_2, \ldots, y_n)\right) \\ + \cdots + \arctan\left(y_n f(y_1, y_2, \ldots, y_n)\right) \end{aligned} \tag{12}$$

for some constant C and all $y_1, y_2, \ldots, y_n > 0$.

Theorem 1. *The only equations of the form (12) satisfied by a nonzero function $f = \sqrt{\frac{p}{q}}$, where p and q are polynomials, are the equations (8)–(11).*

Proof. Letting $x_i = y_i f(y_1, y_2, \ldots, y_n)$, use equations (4) and (12) to write

$$\begin{aligned} D := \tan(C) &= \tan(\arctan(x_1) + \cdots + \arctan(x_n)) \\ &= \frac{e_1(x) - e_3(x) + e_5(x) - \cdots}{e_0(x) - e_2(x) + e_4(x) - \cdots} \\ &= \frac{f e_1(y) - f^3 e_3(y) + f^5 e_5(y) - \cdots}{e_0(y) - f^2 e_2(y) + f^4 e_4(y) - \cdots}. \end{aligned} \tag{13}$$

For the rest of the proof, the y will be suppressed from e_i. Now we break the proof into two cases.

***n* is even.** Let $n = 2m$. Equation (13) can be written as

$$\begin{aligned} &D\left(e_0 - f^2 e_2 + \cdots + (-1)^m f^{2m} e_{2m}\right) \\ &\quad = f e_1 - f^3 e_3 + \cdots + (-1)^{m-1} f^{2m-1} e_{2m-1}. \end{aligned} \tag{14}$$

First, consider the case where $f = p/q$ and p and q are polynomials with no common factors. Equation (14) can be written as

$$\begin{aligned} &D\left(e_0 q^{2m} - p^2 q^{2m-2} e_2 + \cdots + (-1)^{m-1} p^{2m} e_{2m}\right) \\ &\quad = p q^{2m-1} e_1 - p^3 q^{2m-3} e_3 + \cdots + (-1)^{m-1} p^{2m-1} q e_{2m-1}. \end{aligned}$$

Since p is a factor of all but one of the terms, we must have that p divides $e_0 q^{2m}$. This forces $p = \alpha$ for some constant α. Similarly, q is a factor of all but one term, so because the elementary symmetric polynomials are irreducible, either $q = \beta$ or $q = \beta e_{2m}$ for some constant β. The latter option will not work because this produces the term $D e_0 q^{2m} = D(\beta e_{2m})^{2m}$, the only term with degree $(2m)^2$. This implies that f is a constant, necessarily the trivial case $f = 0$.

Now suppose that $f = \sqrt{p/q}$, f is not rational, and p and q are polynomials with no common factors. Equation (14) can be written as

$$\begin{aligned} &D\left(e_0 q^m - p q^{m-1} e_2 + \cdots + (-1)^m p^m e_{2m}\right) \\ &\quad = \sqrt{\frac{p}{q}}\left(q^m e_1 - p q^{m-1} e_3 + \cdots + (-1)^{m-1} p^{m-1} q e_{2m-1}.\right) \end{aligned}$$

This equation only holds in two cases: $D = 0$ or $D = \infty$.

When $D = 0$, we obtain

$$0 = q^{m-1}e_1 - pq^{m-2}e_3 + \cdots + (-1)^{m-1}p^{m-1}e_{2m-1}. \tag{15}$$

As before, the irreducibility of e_k implies there is a constant α such that $p = \alpha$ or $p = \alpha e_1$, and subsequently, there exists a constant β and $k \in \mathbb{N} \cup \{0\}$ such that $q = \beta e_1^k e_{2m-1}$. By replacing p and q in equation (15) and considering the degree of each term, we see that the only possible option (up to scaling) is $m = 2$ ($n = 4$), $p = e_1$, and $q = e_3$. This is equation (11). The case $D = \infty$ is similar, and implies

$$0 = e_0 q^m - pq^{m-1}e_2 + \cdots + (-1)^m p^m e_{2m}.$$

This forces $p = \alpha$ for some constant α and $q = \beta$ or $q = \beta e_{2m}$ for some constant β. The only feasible option (up to scaling) is $m = 1$ ($n = 2$), $p = 1$, and $q = e_2$. This is equation (8).

***n* is odd.** Let $n = 2m + 1$. Equation (13) can be written as

$$D\left(e_0 - f^2 e_2 + \cdots + (-1)^m f^{2m} e_{2m}\right) = f e_1 - f^3 e_3 + \cdots + (-1)^m f^{2m+1} e_{2m+1}.$$

Paralleling the even case, one can show that there is no nontrivial rational function f satisfying this equation. If f takes the form $\sqrt{p/q}$, one again needs to consider two cases, $D = 0$ and $D = \infty$.

If $D = 0$, this forces $p = \alpha$ or $p = \alpha e_1$ for some constant α, and $q = \beta$ or $q = \beta e_{2m+1}$ for some constant β. The only feasible solution (up to scaling) is $m = 1$ ($n = 3$), $p = e_1$, and $q = e_3$. This is equation (9). If $D = \infty$, one has $p = \alpha$ for some constant α and either $q = \beta$ or $q = \beta e_{2m}$ for some constant β. The only feasible solution (up to scaling) is $m = 1$ ($n = 3$), $p = 1$, and $q = e_2$. This is equation (10). ∎

There are other choices of f, but they get more complicated. A simple example uses equation (5) with $\theta_1 + \theta_2 = \pi/4$, producing

$$\frac{\pi}{4} = \arctan\left(\frac{\sqrt{y_1^2 + 6y_1y_2 + y_2^2} - y_1 - y_2}{2y_1}\right) + \arctan\left(\frac{\sqrt{y_1^2 + 6y_1y_2 + y_2^2} - y_1 - y_2}{2y_2}\right)$$

for $y_1, y_2 > 0$.

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