Families of Solutions of a Cubic Diophantine Equation

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This study started with an unusual advertisement which appeared (January, 6, 1996) in *The Globe and Mail*, Canada's national newspaper. Vivikanand Kadarnauth (of Toronto) presented the "first few cases" in a family of solutions to the "cubic version of the Pythagorean equation"

$$a^3 + b^3 + c^3 = d^3 \tag{1}$$

as

$$4^3 + 5^3 + 3^3 = 6^3, \quad 4^3 + 17^3 + 22^3 = 25^3$$

$$16^3 + 23^3 + 41^3 = 44^3, \quad 16^3 + 47^3 + 108^3 = 111^3$$

$$64^3 + 107^3 + 405^3 = 408^3, \quad 64^3 + 155^3 + 664^3 = 667^3$$

Mr. Kadarnauth then asked the reader to find the general pattern. Some of the patterns indicate that the the general solution is

$$(a,b,c,d) = (2^{2m}, 2\cdot 2^{2m} - 3\cdot 2^m + 3, 2^{3m} - 2\cdot 2^{2m} + 3\cdot 2^m - 3, 2^{3m} - 2\cdot 2^{2m} + 3\cdot 2^m)$$

and

$$(a,b,c,d) = (2^{2m},\ 2\cdot 2^{2m} + 3\cdot 2^m + 3, 2^{3m} + 2\cdot 2^{2m} + 3\cdot 2^m, 2^{3m} + 2\cdot 2^{2m} + 3\cdot 2^m + 3).$$

where m varies over the postive integers. One may generalize this by replacing 2^m with x, thus yielding the one-parameter polynomial families of solutions

$$(a,b,c,d) = (x^2, 2x^2 - 3x + 3, x^3 - 2x^2 + 3x - 3, x^3 - 2x^2 + 3x) \tag{2}$$

and

$$(a, b, c, d) = (x^2, 2x^2 + 3x + 3, x^3 + 2x^2 + 3x, x^3 + 2x^2 + 3x + 3). \quad (3)$$

The second family (3) is equivalent to (2). This is seen by replacing x with $-x$ in (2) and re-arranging the terms since $a^3 + b^3 + c^3 = d^3$ implies $a^3 + b^3 + (-d)^3 = (-c)^3$. By letting $x = v/u$ in (3) and multiplying by u^3 gives the family of solutions listed by Jandasek (see [3, p.559]):

$$(a, b, c, d) = (uv^2, 3u^2v+2uv^2+v^3, 3u^3+3u^2v+2uv^2, 3u^3+3u^2v+2uv^2+v^3) \quad (4)$$

The cubic Diophantine equation (1) has been studied for over 400 years. In 1591, P. Bungus (see [3, p.550]) found the smallest positive solution mentioned above, namely

$$4^3 + 5^3 + 3^3 = 6^3, \quad (5)$$

the same year that Vieta found a family of solutions. (Perelman[6, p.139] writes: "It is said that [equation (5)] highly intrigued Plato") Almost 200 years later, Euler (see [3, p.552]) found the general **rational** solution to equation (1) may be represented [5, p.292] as

$$\begin{aligned}(a, b, c, d) \quad = \quad & \big(\sigma\left(-(\xi - 3\eta)(\xi^2 + 3\eta^2) + 1\right), \\ & \sigma\left((\xi^2 + 3\eta^2)^2 - (\xi + 3\eta)\right), \\ & \sigma\left((\xi + 3\eta)(\xi^2 + 3\eta^2) - 1\right), \\ & \sigma\left((\xi^2 + 3\eta^2)^2 - (\xi - 3\eta)\right)\big)\end{aligned} \quad (6)$$

where σ, ξ, η are rationals. The variable σ is simply a scaling factor reflecting the homogeneity of the equation (1). Ramanujan (see [2]) also gave a general solution as

$$(a, b, c, d) = (\alpha + \lambda^2\gamma, \lambda\beta + \gamma, -\lambda\alpha - \gamma, \beta + \lambda^2\gamma) \quad (7)$$

whenever $\alpha^2 + \alpha\beta + \beta^2 = 3\lambda\gamma^2$ (Ramanujan's result was slightly pre-dated by a very similar general solution due to Schwering (see [3, p.557])).

Despite these results, however, there is no known formula characterizing the **integral** solutions to equation (1). In this light, considering various families of solutions is of value. This paper categorizes and extends various families of solutions to equation (1). Many of the results may be found in Dickson[3] and Barbeau[1].

There are many other one-parameter families of solutions to equation (1) besides (2)–(3). Examples are

$$\begin{aligned}(a,b,c,d) = (&(2x-1)(2x^3-6x^2-1), (x+1)(5x^3-9x^2+3x-1),\\ &3x(x+1)(x^2-x+1), 3x(2x-1)(x^2-x+1)) \end{aligned} \tag{8}$$

$$(a,b,c,d) = (x^3+1, 2x^3-1, x(x^3-2), x(x^3+1)) \tag{9}$$

$$(a,b,c,d) = (3x^2, 6x^2 \pm 3x + 1, 3x(3x^2 \pm 2x + 1), c+1) \tag{10}$$

As before, one may let x be a rational number v/u and multiply through by an appropriate power of u to obtain a two-parameter family of **integral** solutions.

A strikingly dissimilar one-parameter family of solutions is due to Ramanujan. Letting

$$\frac{1+53x+9x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} a_n x^n$$

$$\frac{2-26x-12x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} b_n x^n$$

$$\frac{2+8x-10x^2}{1-82x-82x^2+x^3} = \sum_{n\geq 0} c_n x^n$$

yields

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

This result produces "near misses" when considering Fermat's Last Theorem. Hirschhorn[4] has observed that Ramanujan's solutions are contained in

$$(a,b,c,d) = (u^2+7uv-9v^2, -u^2+9uv+v^2, 2u^2-4uv+12v^2, 2u^2+10v^2). \tag{11}$$

Some authors have given two-parameter families of solutions to equation (1) which could have been generated from a one-parameter solution (as we have done earlier). Examples are

$$\begin{aligned}(a,b,c,d) &= (3u^2+5uv-5v^2, 4u^2-4uv+6v^2,\\ &\quad 5u^2-5uv-3v^2, 6u^2-4uv+4v^2)\end{aligned} \tag{12}$$

$$\begin{aligned}(a,b,c,d) &= (3u^2+16uv-7v^2, 6u^2-4uv+14v^2,\\ &\quad -3u^2+16uv+7v^2, 6u^2+4uv+14v^2)\end{aligned} \tag{13}$$

Two-parameter solutions of (1) which do not arise from one-parameter solutions are not so plentiful. Ramanujan (see [1, p.35,48]) discovered

$$\begin{aligned}(a,b,c,d) &= (u^7-3(v+1)u^4+(3v^2+6v+2)u,\\ &\quad 2u^6-3(2v+1)u^3+3v^2+3v+1,\\ &\quad u^6-3v^2-3v-1, u^7-3vu^4+(3v^2-1)u).\end{aligned} \tag{14}$$

In comparing the different families of solutions previously mentioned, one notices that the coefficients in the solution represented by (12) are the same as the values in equation (5). This generalizes to

Theorem 1 *If*

$$a^3+b^3+c^3=d^3 \tag{15}$$

and

$$c(c^2-a^2)=b(d^2-b^2), \tag{16}$$

then

$$(ax^2+cx-c)^3+(bx^2-bx+d)^3+(cx^2-cx-a)^3=(dx^2-bx+b)^3.$$

This theorem may be proved directly by an expansion. It shows that a one-parameter family of solutions may sometimes be constructed from one solution. The next theorem shows exactly where Theorem 1 applies.

Theorem 2 *The only solutions of equations (15)–(16) are*

1. *(trivial) solutions of the form* $(a,b,c,d) = (a,b,-a,b)$
2. *(scaled) solutions of the one-parameter system represented by (9), namely*

$$(a,b,c,d) = (1+u^3, u^4-2u, 2u^3-1, u^4+u).$$

Proof: Substituting Euler's general solution (6) of (15) into (16) gives (after dividing by σ^3)

$$0 = 36\eta^2(\xi-\eta)(\xi^2+3\eta^2-1)(\xi^4+6\xi^2\eta^2+\xi^2+9\eta^4+3\eta^2+1).$$

If $\eta = 0$ or $\xi^2+3\eta^2-1=0$, one falls into the first class of solutions. The only other possibility is if $\xi = \eta$, which yields

$$(a,b,c,d) = (8\eta^3+1, 16\eta^4-4\eta, 16\eta^3-1, 16\eta^4+2\eta).$$

Setting $u = 2\eta$ shows that this case falls into the second class of solutions. $\square$

Note that the second class of solutions is the same as (9). This solution is due to Vieta. Combining Theorems 1 and 2 generates a new two-parameter family of solutions to (1), namely

$$\begin{aligned}(a,b,c,d) = ((1+u^3)x^2+(2u^3-1)(x-1), (u^4-2u)x(x-1)+u^4+u,\\ (2u^3-1)x(x-1)-u^3-1, (u^4+u)x^2-(u^4-2u)(x-1)).\end{aligned} \quad (17)$$

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