UNBOUNDED DUCCI SEQUENCES

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1 Introduction

A simple iterative problem attributed to E. Ducci (see [10]) asks about the limiting behavior of finite strings. Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be defined as

$$f(x_1, \ldots, x_n) = (|x_1 - x_2|, |x_2 - x_3|, \ldots, |x_n - x_1|). \tag{1}$$

Let the *zero-string* be a string whose terms are all zero. The following result was first proved by Ciamberlini and Marengoni [4] in 1937.

Theorem 1.1 *Every integer string of length n will iterate to the zero-string in a finite number steps if and only if $n = 2^m$ for some positive integer m.*

Remarkably, this theorem has been restated and reproved (using various techniques) a number of times since then; see [3], [8], [18], [19], [22], [23], [27]. Other references may be found in [17]. This process has been most commonly referred to as a *Ducci* sequence or the *N-number game*.

Other interesting questions and observations concerning the iteration of f have arisen. Lotan[11] showed that every 4-string with *real* entries iterates to the zero-string in finitely many steps except for strings (up to shifts, reflection and scaling) of the form $(1, q, q^2, q^3)$ where $q \doteq 1.839$ is a root of $q^3 - q^2 - q - 1 =$

0. These exceptional strings iterate to the zero-string asymptotically. Work towards this result was also pursued geometrically in [16]. Webb[24] showed that 4-strings may take arbitrarily long to reach the zero-string; if one considers the Tribonacci numbers defined by

$$t_n = t_{n-1} + t_{n-2} + t_{n-3}, \qquad t_0 = 0,\ t_1 = 1,\ t_2 = 1,$$

then 3 iterates of $(t_n, t_{n-1}, t_{n-2}, t_{n-3})$ yields $2(t_{n-2}, t_{n-3}, t_{n-4}, t_{n-5})$. For strings whose length is not a power of two, various results concerning cycle lengths, which strings are in cycles, and the number of iterations needed to reach a cycle can be found in [1], [5], [7], [12], [13], [14], [15]. Associated to the lengths of cycles is the structure of Pascal's triangle; see [2] and [9].

The map f has a *weighting* of $(\underline{1}, -1)$. The first term is underlined to indicate the relative position of the weighting when it is applied to each term. The weighting is *bounded* since the maximum of any string does not increase under iteration. This boundedness plays a significant role in the proofs of Theorem 1.1 since, among other things, it forces the all strings to eventually enter a cycle. This paper is principally concerned with unbounded weightings.

Section 2 yields a general result concerning weightings that admit strings which iterate to the zero-string. Sections 3 and 4 consider the unbounded weighting $(-1, \underline{2}, -1)$, explicitly

$$f(x_1, \ldots, x_n) = (|2x_1 - x_2 - x_n|, |2x_2 - x_3 - x_1|, \ldots, |2x_n - x_1 - x_{n-1}|). \quad (2)$$

Specifically, Section 3 focuses on results akin to Theorem 1.1 for this new weighting, while Section 4 gives an intricate description of the dynamics of 3-strings. Connections are made to circulant matrices, Fibonacci numbers, and circle maps.

2 Other weightings

Let the *sum* of a weighting be the absolute value of the sum of the terms in that weighting. The next result shows the relevance of the sum.

Theorem 2.1 *Let W be a weighting with sum s. For every string length, there exists a non-zero string which iterates under W to the zero-string if and only if $s = 0$.*

Proof: Denote the weighting W by $(\underline{x_1}, x_2, \ldots, x_k)$. A string of length n, with $n > k$, will to iterate to the zero-string if and only if the $n \times n$ circulant matrix

$$\begin{bmatrix} x_1 & x_2 & x_3 & \ldots & x_k & 0 & \ldots & 0 \\ 0 & x_1 & x_2 & \ldots & x_{k-1} & x_k & \ldots & \\ \vdots & & & \vdots & & & & \vdots \\ x_2 & x_3 & & \ldots & & & & x_1 \end{bmatrix}$$

has a determinant of zero. Note that the position of the underline in the weighting does not change the determinant condition since rows in the circulant matrix may be permuted; this effectively changes the position of the underline. The determinant condition occurs if and only if

$$x_1 + x_2\omega + x_3\omega^2 + \ldots + x_n\omega^{n-1} = 0 \tag{3}$$

where ω is an n^{th} root of unity; see Weisstein[25] for the basics of circulant matrices. If $s = 0$, letting $\omega = 1$ satisfies this condition for any $n \geq k$. If $s \neq 0$, let n, $n > k$, be any prime. Since n is prime, $\omega^n - 1 = 0$ is the lowest degree equation the algebraic number ω can satisfy unless $\omega = 1$. However, $\omega = 1$ does not satisfy Equation (3) if $s \neq 0$. □

More can be said with regards to s. A *one-string* – a string with all ones – iterates to itself times a factor of s. If $s = 0$, the one-string iterates immediately to the zero-string. If $s = 1$, the one-string is fixed. Lastly, if $s \geq 2$, the one-string diverges.

There are many integer weightings besides $(\underline{1}, -1)$ which are bounded. Any integer weighting whose terms are restricted to -1, 0, or 1, and where there are never more than 3 consecutive non-zero terms, will be bounded. If zeros are not allowed in the weighting though, only $(\underline{1}, -1)$ is bounded.

It would be interesting to enrich the dynamics beyond what is seen for $(\underline{1}, -1)$ so that both unboundedness *and* terms iterating to the zero-string are present.

Theorem 2.1 implies a weighting with at least three terms must be considered. The weighting $(-1, \underline{2}, -1)$ is both unbounded and has $s = 0$, so this is studied in the following 2 sections.

3 General Limiting Behavior for $(-1, \underline{2}, -1)$

The first result proves the negative counterpart of Theorem 1.1.

Theorem 3.1 *For any integer $n \neq 2^m$, there are n-strings which do not iterate to the zero-string.*

Proof: The proof is similar in spirit to that found in [3]. Suppose first that the string has odd length. After factoring out all twos, consider its entries mod 2, so that at least one term equals one. The table below shows how the middle term of a substring of three numbers mod 2 iterates:

3-string (mod 2)	*iterate of middle term (mod 2)*
0 0 0	0
0 0 1	1
0 1 0	0
1 0 0	1
1 0 1	0
1 1 0	1
0 1 1	1
1 1 1	0

The only non-trivial predecessor of the zero-string is the one-string mod 2. Using the table, one may show that any predecessor must contain the substring 0011 repeated finitely many times, hence the string length must be a multiple of 4, a contradiction. Therefore, there exist strings of any odd length which do not iterate to the zero-string.

If the string length is n and $p|n$ for some odd prime p, construct a new string by concatenating n/p copies of a string of length p which does not iterate to the zero-string. This new string will also not iterate to the zero-string. $\square$

It remains now to consider only those strings whose string length is a power of two. Unfortunately, Theorem 1.1 for the $(-1, \underline{2}, -1)$ weighting is false: after 24 iterations, the 8-string

$$(1, 2, 3, 0, 1, 0, 1, 2)$$

maps to

$$2^8(1, 2, 3, 0, 1, 0, 1, 2),$$

so this string diverges. After 240 iterations, the 16-string

$$(2, 1, 1, 1, 0, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1)$$

maps to

$$2^{40}(2, 1, 1, 1, 0, 1, 2, 1, 1, 1, 1, 1, 1, 1, 2, 1),$$

so this string also diverges. It is curious to note that though the terms in these strings are small, the strings still diverge. Surprisingly, we will show that, in spite of these counter-examples to our desired result, every 4-strings converges to the zero-string. For this we will need the following lemma.

Lemma 3.1 *Let a, b, c, d be integers. Any string of the form $(0, b, d, d)$, $(0, 0, c, d)$, $(a, 0, c, 2a)$, (a, a, c, c), or (a, b, a, b) iterates to the zero-string.*

The proof is simple and left to the reader.

Theorem 3.2 *All integer 4-strings iterate to the zero-string.*

This theorem was first proved by Robb[20]; the proof below is a streamlined version of his approach.

Proof: The idea behind the proof is straightforward. We show that the maximum of any 4-string no more than doubles after two iterations. Since after two iterations one may always divide out a factor of two, the factored maximum

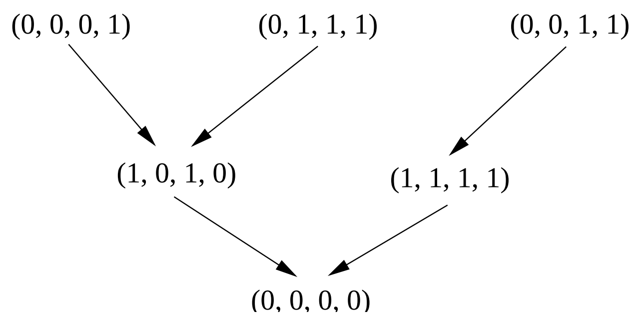

Figure 1: Iterates of a 4-string mod 2

never increases. Lastly we dispose of the cases where the factored maximum does not decrease.

The fact that a two may be factored out after two iterations is simple. Figure 1 shows how 4-strings iterate mod 2. Note that all possible 4-strings mod 2 are represented in the figure if one allows for shifts.

Having outlined the proof, many messy details emerge. There is no obvious way to generalize the approaches used in proving Theorem 1.1, so this proof breaks down into many subcases. Essential details are recorded in tables.

After one iteration of any string, all the entries become non-negative, so let us assume this is the case to begin with. There are two general cases to consider. Starting with the string (a, b, c, d), we first consider a to be the minimum value of the string and d the maximum. The following table summarizes the dynamics after two iterations. In all four subcases, the corresponding inequalities may be used to show that after two iterations the maximum never exceeds $2d$. The third column records those initial strings where after two iterations one obtains $2d$ as a maximum, a "borderline situation".

case	*string after 2 iterates*	*borderline situations*
$2b-a-c\geq 0$ $2c-b-d\geq 0$	$(b+d-2a, \lvert 2b-a-c\rvert,$ $\lvert 2c-b-d\rvert, 2d-a-c)$	$(0,b,d,d)$ (0,0,0,d)
$2b-a-c\geq 0$ $2c-b-d\leq 0$	$(2c-2a, 2d-2b,$ $2c-2a, 2d-2b)$	(a,b,a,b)
$2b-a-c\leq 0$ $2c-b-d\geq 0$	$(4b-4a, 4b-4a,$ $4d-4c, 4d-4c)$	(a,a,c,c)
$2b-a-c\leq 0$ $2c-b-d\leq 0$	$(4b-4a, \lvert 4a-6b+4c-2d\rvert,$ $\lvert 4b-4c\rvert, 2d-2b)$	$(0,0,c,d)$

All the borderline cases are settled wth Lemma 3.1.

The other general case has b as the minimum and d as the maximum. The following table illustrates the dynamics.

case	*string after 2 iterates*	*borderline situations* *and reductions*
$2a-b-d\geq 0$ $2c-b-d\geq 0$	$(4d-4a, 2d-2b,$ $4d-4c, \lvert 6d-4a-4c+2b\rvert)$	$(a,0,c,2a)$ $(a,0,c,2c)$ $(a,0,a,a)$ $b=0$, maximum in term 2
$2a-b-d\geq 0$ $2c-b-d\leq 0$	$(4d-4a, 4c-4b,$ $4c-4b, 4d-4a)$	(a,b,b,a)
$2a-b-d\leq 0$ $2c-b-d\geq 0$	$(4a-4b, 4a-4b,$ $4d-4c, 4d-4c)$	(a,a,c,c)
$2a-b-d\leq 0$ $2c-b-d\leq 0$	$(4a-4b, \lvert 4a+4c-6b-2d\rvert,$ $4c-4b, 2d-2b)$	$(a,0,c,2a)$ $(a,0,c,2c)$ $(a,0,a,a)$ $b=0$, maximum in term 4

As in the first table, all the subcases may be disposed of, with exceptions where one has "$b=0$, maximum in term 2 or 4". To handle these exceptions, note that the only way such strings will *not* have a maximum which decreases is to perpetually fall into these exceptions. In the "term 4" subcase, this forces the second term to be zero, hence using $b = 0$ implies $d = 2a + 2c$. This implies that after two iterations the original string (a, b, c, d) becomes $(4a, 0, 4c, 4a+4c)$, so a four may be factored out, and the factored maximum has decreased. The case with "term 2" may be handled similarly. □

In attempting to duplicate Lotan's earlier-mentioned result (see [11]) for $(-1, \underline{2}, -1)$ weighting, another surprise occurs. The real 4-string

$$\left(1, \frac{1+\sqrt{5}}{2}, 2+\sqrt{5}, \frac{1+\sqrt{5}}{2}\right)$$

maps to

$$(\sqrt{5}-1)\left(1, \frac{1+\sqrt{5}}{2}, 2+\sqrt{5}, \frac{1+\sqrt{5}}{2}\right),$$

so this string is divergent. This is stupefying in light of Theorem 3.2. That theorem implies that any rational 4-string iterates to the zero-string, but now a string with the simplest kind of irrational entries diverges. These results highlight the fact that studying the dynamics of this map on $\mathbb{R}^4$ does not give full insight into what may occur on a subspace, even if that subspace is dense.

Moreover, the appearance of the golden ratio suggests certain strings with Fibonacci numbers are invariant. Letting F_n denote the n^{th} Fibonacci number, it is easy to show that the string $(F_n, F_{n+1}, F_{n+3}, F_{n+1})$ maps to $2\,(F_{n-1}, F_n, F_{n+2}, F_n)$. Like the result of Webb[24] for the weighting $(\underline{1}, -1)$, 4-strings may take arbitrarily long to reach the zero-string. Note that for sufficiently large n the maximum term increases since two exceeds the golden ratio. We then have that the string's (unfactored) maximum will grow in size almost until its last iteration before it collapses to the zero-string.

4 Strings of Length 3

Since the weighting $(-1, \underline{2}, -1)$ is unbounded, it is unclear whether a given string will eventually diverge, cycle, or iterate to the zero-string. For example, a few iterates show that

$$(5,1,4) \to (5,7,2) \to (1,7,8) \to (13,5,8) \to (13,11,2)$$
$$\to (13,7,20) \to (1,19,20) \to (37,17,20) \to (37,23,14) \to (37,5,32) \to \dots$$

This section studies this question for strings of length three. The first result shows the possibilities.

Theorem 4.1

1. *Any integer 3-string maps to $(a, b, a+b)$ (or a shifted version thereof) after one iteration.*
2. *The iterate of any string $(a, b, a+b)$ where $a, b \in \mathbf{Z}^+$ and gcd$(a, b) = 1$, may only have 3 as a common factor.*
3. *Any integer 3-string either diverges or iterates to a fixed point $(x, x, 2x)$ (or shifted version theoreof).*

Proof: Part (1) is trivial and left to the reader. For Part (2), note that $(a, b, a+b)$ maps to $(2b-a, |2a-b|, a+b)$. If this string has a common factor of k, then $2b - a \equiv 0 \bmod k$ and $2a - b \equiv 0 \bmod k$. Adding these implies $3a \equiv 0 \bmod k$ and $3b \equiv 0 \bmod k$, giving the desired result.

To prove Part (3), the key is to observe that the middle term (middle in size, not position) is non-decreasing under iteration. Without loss of generality, let $b \geq a$ so b is the middle term of $(a, b, a+b)$. Since this string maps to $(2b-a, |2a-b|, a+b)$, and $2b - a \geq b$ and $a + b \geq b$, the middle is non-decreasing and remains unchanged if and only $a = b$. This completes the proof. □

Part (2) of Theorem 4.1 is helpful for computations; one need only look for threes to factor out. The more challenging question now is: How can we determine whether a string will diverge or not? The next result provides the basis for an answer.

Theorem 4.2 *Let $a, b \in \mathbf{Z}^+$ with $gcd(a,b) = 1$ and $a \leq b$. Then if $a \not\equiv 1$ mod 3 or $b \not\equiv 2$ mod 3, the string $(a, b, a+b)$ diverges.*

Proof: Scale the string $(a, b, a+b)$ so that it is of the form $(m, 1, 1+m)$, where $m \in [0,1]$. The ratio of the smallest term to the middle term is m. If $m \in [0, 1/2]$, this string iterates to $(2-m, 1-2m, m+1)$, so the ratio of the smallest term to the middle term is $\frac{1-2m}{m+1}$. If $m \in [1/2, 1]$, this string iterates to $(2-m, 2m-1, m+1)$ and the ratio of the smallest term to the middle term is $\frac{2m-1}{2-m}$. In summary, the ratio of the smallest term to the middle term iterates via the map $\bar{f} : [0,1] \to [0,1]$ defined by

$$\bar{f}(x) = \begin{cases} (1-2x)/(1+x), & x \in [0, 1/2] \\ (2x-1)/(2-x), & x \in (1/2, 1]. \end{cases}$$

By part (3) of Theorem 4.1, the string $(a, b, a+b)$ will diverge if and only a/b does not iterate to $x = 1$ under $\bar{f}$ in finitely many steps. The $\bar{f}$-predecessors of any $x \in [0,1]$ are

$$\frac{1-x}{x+2} \quad \text{and} \quad \frac{1+2x}{x+2}.$$

If

$$x = \frac{a}{b}, \quad a, b \in \mathbf{Z}^+, \quad a \equiv 1 \text{ mod } 3, \quad b \equiv 2 \text{ mod } 3$$

this forces the numerator and denominator of the predecessors to have the same form. Since $\bar{f}^{-1}(1) = 0$ or 1 and $\bar{f}^{-1}(0) = 1/2$, we have $\bar{f}^{(k)}(x) = 1$ for some $k \in \mathbf{Z}^+$ only if $x = 0$, $x = 1$, or

$$x = \frac{a}{b}, \quad a, b \in \mathbf{Z}^+, \quad a \equiv 1 \text{ mod } 3, \quad b \equiv 2 \text{ mod } 3,$$

□

Theorem 4.2 implicitly gives the algorithm one may use to determine if a 3-string will diverge or not. After dividing out any factors of 3 from an integer string, if either $a \not\equiv 1$ mod 3 or $b \not\equiv 2$ mod 3, the string will diverge. Otherwise, iterate the string and repeat the same steps.

Since $\bar{f}$ is a 2:1 function on $[0,1] \backslash \{1/2\}$, the complete predecessor set of $x = 1$ is countable. If we consider strings with real entries, this implies that almost

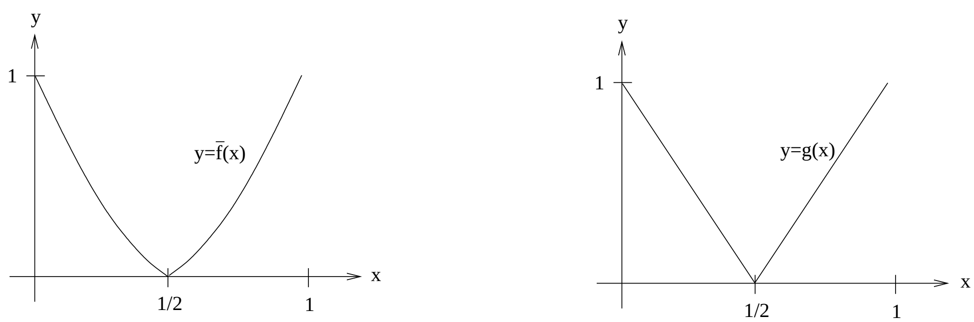

Figure 2: The maps $y = \bar{f}(x)$ and $y = g(x)$.

every 3-string will diverge, and if a/b is irrational, the 3-string will diverge. One may say more about this predecessor set of $x = 1$. Since $|\bar{f}'(x)| > 1$ on $[0, 1] \setminus \{1/2\}$, $\bar{f}$ is an expanding circle map. By an important theorem due to M. Shub (see [6] or [21]), the function $\bar{f}$ is topologically conjugate via a nonsmooth homeomorphism h to the piecewise linear map $g(x) = |1 - 2x|$; see Figure 2. The conjugacy h satisfies $h(1) = 1$. The predecessor set of the fixed point $x = 1$ under the map g is the dense set

$$\left\{ \frac{k}{2^n} : \ 0 \leq k \leq 2^n, \ n = 0, 1, 2, \ldots \right\}$$

This implies the predecessor set for the fixed point $x = 1$ under the map f is also dense on $[0, 1]$. Therefore, though an arbitrary 3-string will almost always diverge, it is arbitrarily close to one which iterates to a fixed point.

5 Conclusion

The previous sections have demonstrated how rich the dynamics of unbounded Ducci sequences can be with the $(-1, \underline{2}, -1)$ weighting. Many questions remained to be addressed:

Question 1: What are the dynamics of bounded integer weightings besides $(\underline{1}, -1)$?

Some examples here are obvious, such as $(\underline{1}, 0 - 1)$ on any $2n$-string; the dynamics are the same as considering two n-strings with the $(\underline{1}, -1)$. However, the dynamics of the weighting $(\underline{1}, 0, 0, -1)$ on 8-strings is far from clear.

Question 2: Are there unbounded integer weightings with a sum of zero which have no non-zero cycles?

A candidate for such a weighting is $(-1, \underline{3}, 2)$. Numerical evidence suggests that all 4-strings iterate eventually either to the zero-string or to some multiple of $(5, 1, 1, 7)$, which diverges.

Question 3: What are the dynamics of weightings with rational terms? real terms?

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