A NEW REPRESENTATION FOR LEGENDRE POLYNOMIALS

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ABSTRACT. By identifying the terms in the LU decomposition of an appropriate matrix, a new representation for Legendre polynomials is found.

In [1], Chamberland uses the LU decomposition of matrices, a tool typically used in numerical linear algebra, to discover and prove combinatorial identities. Specifically, take a highly structured square matrix, compute the LU decomposition, identify the terms in both \( L \) and \( U \), and thus produce a conjectured sum formula. To see the patterns in \( L \) and \( U \), one usually needs to consider an \( n \times n \) matrix where \( n \) is sufficiently large. Sequence recognition is supported by using the On-Line Encyclopedia of Integer Sequences (http://oeis.org/) or the Maple package \texttt{gfun}.

The goal of this paper is to use the LU decomposition process to discover and prove a new representation for the Legendre polynomials. Identities and properties of these polynomials are ubiquitous in the literature[3]. A standard way to define the Legendre polynomials is with its Rodrigue’s representation:

\[
P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}[(x^2 - 1)^n]
\]  

where \( n \) is a natural number. Another approach is to generate these polynomials from the recurrence relationship

\[
(n + 2)P_{n+2}(x) = (2n + 3)xP_{n+1}(x) - (n + 1)P_n(x)
\]  

coupled with \( P_0(x) = 1 \) and \( P_1(x) = x \). Both of these characterizations play a role in the ensuing analysis.

Inspired by the Rodrigue representation (0.1), construct an \( n \times n \) matrix \( M \) whose \((i, j)\) entry is

\[
M_{ij} = \frac{d^{i-1}}{dx^{i-1}}[(x^2 - 1)^{j-1}]
\]

The LU factorization, performed with Maple, produces the following when \( n = 4 \):

\[
\begin{bmatrix}
1 & x^2 - 1 & (x^2 - 1)^2 & (x^2 - 1)^3 \\
0 & 2x & 4x(x^2 - 1) & 6x(x^2 - 1)^2 \\
0 & 2 & 12x^2 - 4 & 6(x^2 - 1)(5x^2 - 1) \\
0 & 0 & 24x & 120x^3 - 72x
\end{bmatrix}
= \begin{bmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
0 & 1/x & 1 & 0 \\
0 & 0 & 3/x & 1
\end{bmatrix}
\begin{bmatrix}
1 & x^2 - 1 & (x^2 - 1)^2 & (x^2 - 1)^3 \\
0 & 2x & 4x(x^2 - 1) & 6x(x^2 - 1)^2 \\
0 & 0 & 8x^2 & 24x^2(x^2 - 1) \\
0 & 0 & 0 & 384x^4
\end{bmatrix}
\]
By choosing larger values of \( n \) and looking for a pattern, one eventually conjectures forms for the \((i, j)\) entry of both \(L\) and \(U\):

\[
L_{ij} = \begin{cases} 
(2i-2)!((i-1)!!)^{i-j}x^j, & i \geq j, \\
0, & \text{otherwise,}
\end{cases}
\]

and

\[
U_{ij} = \begin{cases} 
\frac{(j-1)!!}{(i-1)!!}(x^2 - 1)^j x^{i-1}, & i \leq j, \\
0, & \text{otherwise.}
\end{cases}
\]

Since \(M_{i,j} = \min(i,j)\sum_{k=1}^{\min(i,j)} L_{i,k}U_{k,j}\), this leaves us with the conjecture (after some simplification)

\[
\frac{d^k}{dx^k}(x^2 - 1)^j = \sum_{k=0}^{\min(i,j)} \frac{j!(2i - 2k)!}{(i-k)!(j-k)!} \binom{i}{2i-2k}(2x)^{2k-i}(x^2 - 1)^j j - k 
\tag{0.3}
\]

Since our goal is to find a representation for Legendre polynomials, we are not interested in proving this formula in its full generality, but only in the special case \(i = j\). Coupling this observation with Rodrigue’s representation (0.1) suggests that we consider the polynomial expressions

\[
f_j := \frac{1}{j!2^j} \sum_{k=0}^{j} \frac{j!(2j - 2k)!}{(j-k)!2^j((j-k)!!)^2} \binom{j}{2j-2k}(2x)^{2k-j}(x^2 - 1)^j j - k
\]

It is possible, albeit cumbersome, to prove that \(f_j\) is the \(j\)th Legendre polynomial by using known identities. However, this approach can be avoided by using Zeilberger’s algorithm (see [2]) for combinatorial sums. Given a sum of hypergeometric type, this technique produces a recurrence relation satisfied by the sum. Using Maple’s built-in command for Zeilberger’s algorithm, one finds that

\[
(j + 2)f_{j+2}(x) = (2j + 3)xP_{j+1}(x) - (j + 1)P_{j}(x)
\]

for all natural numbers \(j\), the same recurrence as equation (0.2). It is easy to see that \(f_0 = 1\) and \(f_1 = x\), implying that the expression \(f_j\) is indeed the \(j\)th Legendre polynomial, that is,

\[
P_j(x) = \sum_{k=0}^{j} \frac{(2j - 2k)!}{2^j((j-k)!!)^2} \binom{j}{2j-2k}(2x)^{2k-j}(x^2 - 1)^j j - k 
\tag{0.4}
\]

This new expression can be compared to two similar well-known expressions [3]:

\[
P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} \binom{n}{k}^2 (x - 1)^{n-k}(x + 1)^k 
\tag{0.5}
\]

and

\[
P_n(x) = \frac{1}{2^n} \sum_{k=0}^{n} (-1)^k \binom{n}{k} \binom{2n - 2k}{k} x^{n-2k} 
\tag{0.6}
\]

The first formula is readily expanded around \(x = \pm 1\), while the second formula expands around \(x = 0\). The new formula (0.4) can be expanded around all three values.
HYPERGEOMETRIC TEMPLATE

REFERENCES


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