Unbounded Orbits and Binary Digits

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On August 6, 2000 at UCLA, Ron Graham gave a joint address for the American Mathematical Society and the Mathematical Association of America entitled "Mathematics in the 21$^{\text{st}}$ Century: Problems and Prospects." His talk included the following open problem:

Consider the sequence $\{x_n\}_{n=0}^{\infty}$ generated by

$$x_{n+1} = x_n - \frac{1}{x_n}, \quad x_0 = 2.$$

Question: Is the sequence $\{x_n\}$ unbounded?

Graham offered some modest numerical evidence that the sequence is unbounded, but stated that a proof seems elusive. This note will show more precisely why this is such a hard question.

A natural approach is to consider the structure of the periodic orbits of $f(x) = x - 1/x$. Since this map is clearly expanding, no periodic orbit is locally attracting. The following result also excludes rational orbits from being eventually periodic.

Theorem 1.1 *If $x \notin \{-1, 0, 1\}$ is a rational number, then the orbit of x is neither eventually periodic nor iterates to 0.*

Proof: The map f is a 2:1 function. Note that

$$f\left(\frac{a}{b}\right) = f\left(\frac{b}{-a}\right) = \frac{(a-b)(a+b)}{ab}. \tag{1}$$

If $a, b \in \mathbf{Z}$ and $(a, b) = 1$, then $((a-b)(a+b), ab) = 1$. Expressing all rational numbers in lowest form, this implies that if a rational number r has a rational predecessor, the denominator of the predecessor must be no greater than the denominator of r, with equality only if r is an integer or if $r = 1/m$ for some integer m. However, if $r \in \mathbf{Z}$, then

$$r \overset{f}{\leftarrow} \frac{r \pm \sqrt{r^2+4}}{2}.$$

The right-hand expression is rational only if $r = 0$, but

$$0 \overset{f}{\leftarrow} \pm 1 \overset{f}{\leftarrow} \frac{\pm 1 \pm \sqrt{5}}{2}.$$

If $r = 1/m$ for some integer m, then

$$r \overset{f}{\leftarrow} \frac{1 \pm \sqrt{1+4m^2}}{2m}.$$

The right-hand expression is never rational. In conclusion, since the predecessor path of every rational number is eventually an irrational number, the set of rational numbers is forward invariant, and $\{-1, 0, 1\}$ are the only rationals iterating in finitely many steps to infinity, we have the desired result. $\square$

It should be noted that this proof may be used to settle Problem B4 on the recently held 62^{nd} William Lowell Putnam Mathematical Competition, December 2001. Specifically, if S denotes the set of rationals different from -1,0 or 1, then

$$\bigcap_{n=1}^{\infty} f^{(n)}(S) = \emptyset.$$

The map f is part of a more general family of functions, namely

$$f_a(x) = a\left(x - \frac{1}{x}\right).$$

The structure of $f_{1/2}$ makes this function more accessible to study.

Theorem 1.2 *The map $f_{1/2}$ is topologically conjugate to the map $2x$ mod 1 on the interval $[0,1)$.*

Proof: Introducing the conjugating function $h(x) = \cot(\pi x)$, we find that f is conjugate to g, where g is defined by

$$\begin{aligned} g(x) &= h^{-1} \circ f \circ h(x) \\ &= \frac{1}{\pi}\cot^{-1}(\cot(2\pi x)) \\ &= 2x \bmod 1. \end{aligned}$$

□

The dynamics of the map $2x$ mod 1 are well-understood; it is equivalent to considering the shift map on the binary form of a number. The standard treatment of symbol sequences shows that the set of periodic orbits is dense and there exist dense orbits, thus our original map $f_{1/2}$ shares such dynamical properties. The same results could be seen by showing that the Julia set of $z \mapsto \frac{1}{2}(z - 1/z)$ is the real-axis and using properties of Julia sets of rational functions; see [1] or [3]. This highlights the sensitivity of the map f and the fact that determining the dynamics of a particular orbit can be much more difficult than finding generic results.

If we modified our original problem to consider iterating $f_{1/2}$ instead of f_1 and used the same initial condition $x_0 = 2$, the conjugacy implies the orbit is dense on the real axis if and only if the binary expansion of $(\cot^{-1} 2)/\pi$ contains substrings with every possible combination of zeros and ones. One may hope that if this number was "sufficiently" irrational, such would be the case. Unfortunately, a number may be transcendental but not satisfy this condition. The simplest example is the Thue-Morse constant, defined by

$$P = \frac{1}{2} \sum_{n=0}^{\infty} P(n) 2^{-n}$$

where the parity $P(n)$ is the number (mod 2) of ones in the binary expansion of n. It is known that P is transcendental yet there are never three consecutive zeros or three consecutive ones; see [4].

Theorem 1.2 may be generalized to f_a for all $a > 0$.

Theorem 1.3 *The map f_a is topologically conjugate to the map $2x$ mod 1 on the interval $[0, 1)$ for each $a > 0$.*

To prove this result, we need an important theorem concerning expanding circle maps. A C^1 map $f : S^1 \to S^1$ is *expanding* if there exist constants $C > 0$ and $\lambda > 1$ such that

$$|Df^n(x)| > C\lambda^n$$

for all $n \in \mathbf{Z}^+$ and all $x \in S^1$. An expanding map is a covering map of degree d, that is, it is a surjective local homeomorphism such that the pre-image of each point consists of exactly d points. The following important result is due to Michael Shub (see [5], [2, p.89]).

Theorem 1.4 *Let $f : S^1 \to S^1$ be an expanding map of degree d. If $g : S^1 \to S^1$ is a covering map of degree d, then there exists a (not necessarily strictly) monotone and surjective map $h : S^1 \to S^1$ such that $h \circ g = f \circ h$.*

In other words, an expanding map of degree d is conjugate to the map $x \to dx \bmod 1$ on $[0,1)$. Now the proof of Theorem 1.3:

Proof: Using the conjugacy $h(x) = \cot(x)$, we define

$$\begin{aligned} g_a(x) &= h^{-1} \circ f_a \circ h(x) \\ &= \cot^{-1}(2a\cot(2x)). \end{aligned}$$

The functions g_a are covering maps of degree 2 for all $a > 0$. Calculating

$$g_a'(x) = \frac{4a}{\sin^2(2x) + 4a^2\cos^2(2x)},$$

one may easily show that $g_a'(x) > 1$ for all $x \in [0,\pi)$ if $a > 1/4$, therefore by Theorem 1.4, these functions are conjugate to $2x$ mod 1 on the interval $[0,1)$. For $a \in (0,1/4]$, we consider $D(g_a(g_a(x)))$. Using

$$\sin(g_a(x)) = \frac{1}{\sqrt{1+4a^2\cot^2(2x)}}, \qquad \cos(g_a(x)) = \frac{2a\cot(2x)}{\sqrt{1+4a^2\cot^2(2x)}},$$

we find that

$$g_a'(g_a(x)) = \frac{a(1+4a^2\cot^2(2x))^2}{4a^2\cot^2(2x) + a^2(4a^2\cot^2(2x)-1)^2}.$$

Letting $y = \cos(2x)$ implies

$$\begin{aligned} D(g_a(g_a(x))) &= g_a'(g_a(x))g_a'(x) \\ &= \frac{4a^2(1-y^2+4a^2y^2)}{4a^2y^2(1-y^2) + a^2(4a^2y^2-1+y^2)^2}. \end{aligned}$$

We wish to show that this expression is greater than 1 for all $y \in [-1,1]$. This is accomplished by showing that the denominator is smaller than the numerator, since

$$\begin{aligned} &4a^2y^2(1-y^2) + a^2(4a^2y^2-1+y^2)^2 - 4a^2(1-y^2+4a^2y^2) \\ &= y^4(a^2(1+4a^2)^2 - 4a^2) + y^2(4a^2 - 2a^2(4a^2+1) - 4a^2(4a^2-1)) - 3a^2 \\ &= a^2\left[y^4(16a^4+8a^2-3) + y^2(-24a^2-2) - 3\right]. \end{aligned}$$

This expression is negative for all $y \in [-1,1]$ and $a \in (0,1/4]$. For each such a there exists some $\lambda > 1$ such that $D(g_a(g_a(x))) \geq \lambda > 1$ for all $x \in [0,\pi)$. Since $g_a'(x) \geq \lambda_0 > 0$, we have for every positive integer k

$$D\left(g_a^{(2k)}\right) \geq \left(\sqrt{\lambda}\right)^{2k}, \quad D\left(g_a^{(2k+1)}\right) \geq \lambda_0 \left(\sqrt{\lambda}\right)^{2k}.$$

This implies

$$D\left(g_a^{(k)}\right) \geq C\left(\sqrt{\lambda}\right)^{k}, \quad C = \min\{1, \lambda_0/\sqrt{\lambda})$$

therefore g_a is expanding. By Theorem 1.4, g_a is conjugate to $2x$ mod 1 on the interval $[0,1)$. □

In summary, solving the original problem presented by Graham involves two serious obstacles. First, though we know that f_1 is conjugate to $2x$ mod 1 on the interval $[0,1)$, finding the conjugating map seems very difficult, if not impossible. Second, even if one can determine this conjugating function, one must still determine the limiting dynamics on the transformed initial condition. This involves understanding the distribution of the binary digits of a given number, another formidable task. It is known, however, that the set of initial points $x \in [0,1)$ whose orbit under $2x$ mod 1 is dense has measure 1, so Theorem 1.3 implies probabilistically that the orbit is dense on the real line.

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References

[1] A.F. Beardon. *Iteration of Rational Functions.* Springer, 1991.

[2] W. de Melo and S. van Strien. *One-Dimensional Dynamics.* Springer, 1993.

[3] N. Steinmetz. *Rational Iteration.* de Gruyter, 1993.

[4] E.W. Weisstein. *CRC Concise Encyclopedia of Mathematics*. Chapman and Hall/CRC, 1999.

[5] M. Shub. *Endomorphisms of compact differentiable manifolds*. *American Journal of Mathematics*, 91:175–199, 1969.