FACTORED MATRICES CAN GENERATE COMBINATORIAL IDENTITIES

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Abstract. By identifying the terms in the LU decomposition of various matrices, one produces combinatorial identities. Examples are given with formulas involving binomial coefficients and other numbers arising from simple recurrence formulas, number-theoretic functions, q-series, and orthogonal polynomials.

1. Introduction

The LU decomposition is a standard matrix factorization which is used in numerical linear algebra. The goal of this paper is to use the LU decomposition to build identities. A simple but illustrative example follows. Consider the matrix whose (i,j)-entry is $(i+j-1)^2$ and construct its LU decomposition:

$$\begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 4 & 9 & 16 & 25 & 36 \\ 9 & 16 & 25 & 36 & 49 \\ 16 & 25 & 36 & 49 & 64 \\ 25 & 36 & 49 & 64 & 81 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 0 & 0 & 0 \\ 9 & \frac{20}{7} & 1 & 0 & 0 \\ 16 & \frac{39}{7} & 3 & 1 & 0 \\ 25 & \frac{64}{7} & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 4 & 9 & 16 & 25 \\ 0 & -7 & -20 & -39 & -64 \\ 0 & 0 & \frac{8}{7} & \frac{24}{7} & \frac{48}{7} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Here we start with a 5×5 matrix, but choosing a larger square matrix does not modify the terms in the 5×5 submatrices.

Studying the terms in the two right-side matrices quickly yields a pattern. The (i,j) entry of L seems to be

$$L_{i,j} = \begin{cases} i^2, & j = 1, \\ (i-1)(3i+1)/7, & j = 2, \\ (i-1)(i-2)/2, & j = 3, \\ 1, & i = j, \\ 0, & \text{otherwise}, \end{cases}$$

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while the (i,j) entry of U appears to be

$$U_{i,j} = \begin{cases} j^2, & i=1, \\ -(j-1)(3j+1), & i=2, \\ 4(i-1)(i-2)/7, & i=3, \\ 0, & i \geq 4. \end{cases}$$

Putting this together suggests the identity

$$(i+j-1)^2 = i^2j^2 - \left(\frac{(i-1)(3i+1)}{7}\right)(-(j-1)(3j+1)) + \left(\frac{(i-1)(i-2)}{2}\right)\left(\frac{4(j-2)(j-1)}{7}\right)$$

which may be rearranged into

$$7(i+j-1)^2 = 7i^2j^2 - (i-1)(3i+1)(j-1)(3j+1) + 2(i-2)(i-1)(j-2)(j-1).$$

This formula may be trivially verified by expanding and simplifying each side.

The example shows the basic approach which will be taken: apply the LU decomposition to a matrix with a particular structure, identify the terms in both L and U, and thus produce a conjectured sum formula. To see the patterns in L and U, one usually needs to consider an $n \times n$ matrix where $n \geq 5$, but sometimes larger values of n are helpful. Sequence recognition is supported by using the On-Line Encyclopedia of Integer Sequences (`http://oeis.org/`) or the Maple package `gfun`.

Proving a conjectured sum is usually straightforward. While a few identities are disposed of using induction or special formulas such as Binet's formula for Fibonacci numbers, the most general tool used to handle the combinatorial sums is Zeilberger's algorithm (see [2]). This algorithm is implemented in Maple and used to provide a computational symbolic proof. In two of the last examples, proofs could not be found; this is explicitly stated in the appropriate places.

In a handful of examples, specifically, those involving Fibonacci numbers or Chebyshev polynomials, the terms satisfy low-degree recurrence relationships, so it is not surprising that the corresponding Hankel matrix has low rank. A source of information on such matrices is [1].

Examples will be generated from sequences encountered with simple recurrence relations, number-theoretic functions, q-series, orthogonal polynomials, and expressions involving sums or products. Some of these formulas could not be found in the literature.

2. Examples from Simple Recurrences

A popular choice involves the Fibonacci numbers, where one finds

$$\begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 \\ 1 & 2 & 3 & 5 & 8 & 13 \\ 2 & 3 & 5 & 8 & 13 & 21 \\ 3 & 5 & 8 & 13 & 21 & 34 \\ 5 & 8 & 13 & 21 & 34 & 55 \\ 8 & 13 & 21 & 34 & 55 & 89 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 & 0 \\ 3 & 2 & 0 & 1 & 0 & 0 \\ 5 & 3 & 0 & 0 & 1 & 0 \\ 8 & 5 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 3 & 5 & 8 \\ 0 & 1 & 1 & 2 & 3 & 5 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

which produces the well-known identity

$$F_{i+j-1} = F_i F_j + F_{i-1} F_{j-1}.$$

The squares of Fibonacci numbers produces

$$\begin{bmatrix} 1 & 1 & 4 & 9 & 25 \\ 1 & 4 & 9 & 25 & 64 \\ 4 & 9 & 25 & 64 & 169 \\ 9 & 25 & 64 & 169 & 441 \\ 25 & 64 & 169 & 441 & 1156 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 4 & \frac{5}{3} & 1 & 0 & 0 \\ 9 & \frac{16}{3} & 2 & 1 & 0 \\ 25 & 13 & 6 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 4 & 9 & 25 \\ 0 & 3 & 5 & 16 & 39 \\ 0 & 0 & \frac{2}{3} & \frac{4}{3} & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Each term may be factored as a product of Fibonacci numbers, revealing a pattern which leads to the identity

$$F_{i+j-1}^2 = F_i^2 F_j^2 + \frac{1}{3} F_{i-1} F_{i+2} F_{j-1} F_{j+2} + \frac{2}{3} F_{i-2} F_{i-1} F_{j-2} F_{j-1}.$$

Higher powers of Fibonacci numbers yield similar, albeit more complicated, results. All the Fibonacci identities may be proven using the familiar Binet formula.

For all the remaining examples in this section, a proof of the conjectured identity follows by using Zeilberger's algorithm.

The Catalan numbers also have a nice representation in this form. Letting the (i, j) entry be $\binom{2(i+j-1)}{i+j-1}/(i+j)$, we find

$$\begin{bmatrix} 1 & 2 & 5 & 14 & 42 \\ 2 & 5 & 14 & 42 & 132 \\ 5 & 14 & 42 & 132 & 429 \\ 14 & 42 & 132 & 429 & 1430 \\ 42 & 132 & 429 & 1430 & 4862 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 \\ 42 & 48 & 27 & 8 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 5 & 14 & 42 \\ 0 & 1 & 4 & 14 & 48 \\ 0 & 0 & 1 & 6 & 27 \\ 0 & 0 & 0 & 1 & 8 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

This is an example of a Cholesky factorization, that is, where $L = U^T$. The (i, j) entry of L is $2j\binom{2i-1}{i-j}/(j+i)$, yielding the identity

$$\binom{2i+2j-2}{i+j-1} \frac{1}{i+j} = \sum_{k=1}^{\min(i,j)} \frac{4k^2}{(k+i)(k+j)} \binom{2i-1}{i-k} \binom{2j-1}{j-k}.$$

By letting $j = i$, one may write every other Catalan number as a sum of squares:

$$\binom{4i-2}{2i-1}\frac{1}{2i} = \sum_{k=1}^{i}\left(\frac{2k}{k+i}\binom{2i-1}{i-k}\right)^2.$$

Trying factorials as entries, one finds

$$\begin{bmatrix} 0! & 1! & 2! & 3! & 4! \\ 1! & 2! & 3! & 4! & 5! \\ 2! & 3! & 4! & 5! & 6! \\ 3! & 4! & 5! & 6! & 7! \\ 4! & 5! & 6! & 7! & 8! \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 2 & 4 & 1 & 0 & 0 \\ 6 & 18 & 9 & 1 & 0 \\ 24 & 96 & 72 & 16 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 & 2 & 6 & 24 \\ 0 & 1 & 4 & 18 & 96 \\ 0 & 0 & 4 & 36 & 288 \\ 0 & 0 & 0 & 36 & 576 \\ 0 & 0 & 0 & 0 & 576 \end{bmatrix}$$

The right-hand entries may be written in terms of binomial coefficients, producing the identity

$$(n+m)! = n!m! \sum_{k=0}^{\min(m,n)} \binom{n}{k}\binom{m}{k}$$

or

$$\binom{m+n}{m} = \sum_{k=0}^{m}\binom{m}{k}\binom{n}{k}, \; m \leq n.$$

This is a special case of Vandermonde's identity.

Modifying the last example, take the (i,j) entry to be $\Gamma(i+j-3/2)/\sqrt{\pi}$. This produces

$$\begin{bmatrix} 1 & 1/2 & 3/4 & \frac{15}{8} & \frac{105}{16} & \frac{945}{32} \\ 1/2 & 3/4 & \frac{15}{8} & \frac{105}{16} & \frac{945}{32} & \frac{10395}{64} \\ 3/4 & \frac{15}{8} & \frac{105}{16} & \frac{945}{32} & \frac{10395}{64} & \frac{135135}{128} \\ \frac{15}{8} & \frac{105}{16} & \frac{945}{32} & \frac{10395}{64} & \frac{135135}{128} & \frac{2027025}{256} \\ \frac{105}{16} & \frac{945}{32} & \frac{10395}{64} & \frac{135135}{128} & \frac{2027025}{256} & \frac{34459425}{512} \\ \frac{945}{32} & \frac{10395}{64} & \frac{135135}{128} & \frac{2027025}{256} & \frac{34459425}{512} & \frac{654729075}{1024} \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/2 & 1 & 0 & 0 & 0 & 0 \\ 3/4 & 3 & 1 & 0 & 0 & 0 \\ \frac{15}{8} & \frac{45}{4} & 15/2 & 1 & 0 & 0 \\ \frac{105}{16} & \frac{105}{2} & \frac{105}{2} & 14 & 1 & 0 \\ \frac{945}{32} & \frac{4725}{16} & \frac{1575}{4} & \frac{315}{2} & \frac{45}{2} & 1 \end{bmatrix}\begin{bmatrix} 1 & 1/2 & 3/4 & \frac{15}{8} & \frac{105}{16} & \frac{945}{32} \\ 0 & 1/2 & 3/2 & \frac{45}{8} & \frac{105}{4} & \frac{4725}{32} \\ 0 & 0 & 3/2 & \frac{45}{4} & \frac{315}{4} & \frac{4725}{8} \\ 0 & 0 & 0 & \frac{45}{4} & \frac{315}{2} & \frac{14175}{8} \\ 0 & 0 & 0 & 0 & \frac{315}{2} & \frac{14175}{4} \\ 0 & 0 & 0 & 0 & 0 & \frac{14175}{4} \end{bmatrix}.$$

The (i,j) entry of L is

$$\frac{(i-1)!}{(i-j)!(j-1)!}\frac{\Gamma(i-1/2)}{\Gamma(j-1/2)}$$

while the (i, j) entry of U is

$$\frac{(j-1)!}{(j-i)!}\frac{\Gamma(j-1/2)}{\sqrt{\pi}}.$$

These combine and simplify to produce

$$\frac{\binom{2i+2j}{2i}}{\binom{i+j}{i}} = \sum_{k=0}^{\min(i,j)} \binom{i}{k}\binom{j}{k}\frac{4^k}{\binom{2k}{k}}.$$

The special case with $i = j$ produces

$$\frac{\binom{4i}{2i}}{\binom{2i}{i}} = \sum_{k=0}^{i} \binom{i}{k}^2 \frac{4^k}{\binom{2k}{k}}.$$

By considering the central binomial coefficients, namely (i, j) entry being $\binom{2(i+j-2)}{i+j-2}$, one finds

$$\begin{bmatrix} 1 & 2 & 6 & 20 & 70 & 252 \\ 2 & 6 & 20 & 70 & 252 & 924 \\ 6 & 20 & 70 & 252 & 924 & 3432 \\ 20 & 70 & 252 & 924 & 3432 & 12870 \\ 70 & 252 & 924 & 3432 & 12870 & 48620 \\ 252 & 924 & 3432 & 12870 & 48620 & 184756 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 \\ 6 & 4 & 1 & 0 & 0 & 0 \\ 20 & 15 & 6 & 1 & 0 & 0 \\ 70 & 56 & 28 & 8 & 1 & 0 \\ 252 & 210 & 120 & 45 & 10 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 6 & 20 & 70 & 252 \\ 0 & 2 & 8 & 30 & 112 & 420 \\ 0 & 0 & 2 & 12 & 56 & 240 \\ 0 & 0 & 0 & 2 & 16 & 90 \\ 0 & 0 & 0 & 0 & 2 & 20 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Some manipulation gives

$$\binom{2i+2j}{i+j} = \binom{2i}{i}\binom{2j}{j} + 2\sum_{k=1}^{\min(i,j)} \binom{2i}{i-k}\binom{2j}{j-k}$$

The special case $i = j$ yields another sum of squares identity:

$$\binom{4i}{2i} = \binom{2i}{i}^2 + 2\sum_{k=1}^{i} \binom{2i}{i-k}^2.$$

Considering the Vandermonde matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 1 & 4 & 9 & 16 & 25 & 36 & 49 \\ 1 & 8 & 27 & 64 & 125 & 216 & 343 \\ 1 & 16 & 81 & 256 & 625 & 1296 & 2401 \\ 1 & 32 & 243 & 1024 & 3125 & 7776 & 16807 \\ 1 & 64 & 729 & 4096 & 15625 & 46656 & 117649 \end{bmatrix},$$

an LU decomposition produces L with terms

$$L_{ij} = \begin{cases} S_2(i,j), & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

where $S_2(i,j)$ are Stirling numbers of the second kind, and the U matrix has terms

$$U_{ij} = \begin{cases} (j-1)!/(i-1)!, & i \leq j, \\ 0, & \text{otherwise}. \end{cases}$$

These combine to give

$$j^{i-1} = \sum_{k=1}^{\min(i,j)} S_2(i,k)\frac{(j-1)!}{(j-k)!}.$$

The Bell numbers are defined as

$$B_n = \sum_{k=1}^{n} S_2(n,k).$$

Letting the (i, j) term take the form B_{i+j-2} produces

$$\begin{bmatrix} 1 & 1 & 2 & 5 & 15 & 52 & 203 \\ 1 & 2 & 5 & 15 & 52 & 203 & 877 \\ 2 & 5 & 15 & 52 & 203 & 877 & 4140 \\ 5 & 15 & 52 & 203 & 877 & 4140 & 21147 \\ 15 & 52 & 203 & 877 & 4140 & 21147 & 115975 \\ 52 & 203 & 877 & 4140 & 21147 & 115975 & 678570 \\ 203 & 877 & 4140 & 21147 & 115975 & 678570 & 4213597 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 5 & 10 & 6 & 1 & 0 & 0 & 0 \\ 15 & 37 & 31 & 10 & 1 & 0 & 0 \\ 52 & 151 & 160 & 75 & 15 & 1 & 0 \\ 203 & 674 & 856 & 520 & 155 & 21 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 2 & 5 & 15 & 52 & 203 \\ 0 & 1 & 3 & 10 & 37 & 151 & 674 \\ 0 & 0 & 2 & 12 & 62 & 320 & 1712 \\ 0 & 0 & 0 & 6 & 60 & 450 & 3120 \\ 0 & 0 & 0 & 0 & 24 & 360 & 3720 \\ 0 & 0 & 0 & 0 & 0 & 120 & 2520 \\ 0 & 0 & 0 & 0 & 0 & 0 & 720 \end{bmatrix}.$$

Some experimentation reveals that the first matrix on the right has (i, j) entries

$$\sum_{k=0}^{i-1} \binom{k}{j-1} S_2(i-1, k),$$

while the second matrix has (i, j) entries

$$(i-1)! \sum_{k=0}^{j-1} \binom{k}{i-1} S_2(j-1, k).$$

Put together, one obtains,

$$B_{i+j} = \sum_{m=0}^{\min(i,j)} m! \left(\sum_{k=0}^{i} \binom{k}{m} S_2(i, k) \right) \left(\sum_{k=0}^{j} \binom{k}{m} S_2(j, k) \right).$$

Note that when $i = j$ one has

$$B_{2i} = \sum_{m=0}^{i} m! \left(\sum_{k=0}^{i} \binom{k}{m} S_2(i, k) \right)^2.$$

This formula is a special case of one found in [3].

Many other similar choices of matrices can be made leading to identities involving powers of numbers and binomial coefficients. All of these may also be proven using Zeilberger's algorithm.

3. Number-theoretic Functions

Applications to number-theoretic functions produce different types of patterns than those previously seen. For example, letting the (i, j) term be $\gcd(i, j)$, one finds

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 3 & 1 & 1 & 3 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ 1 & 1 & 1 & 1 & 5 & 1 \\ 1 & 2 & 3 & 2 & 1 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

The entries of L are

$$L_{ij} = \begin{cases} 1, & j|i \\ 0, & \text{otherwise} \end{cases}$$

while the entries of U are

$$U_{ij} = \begin{cases} \phi(i), & i|j \\ 0, & \text{otherwise} \end{cases}$$

where $\phi(n)$ is the totient function. Putting this together forces

$$\gcd(i, j) = \sum_{k|\gcd(i,j)} \phi(k).$$

Since $\gcd(i, j)$ can take any value, this implies

$$n = \sum_{k|n} \phi(k),$$

a well-known identity. Generalizing $\gcd(i, j)$ to $\gcd(i, j)^s$, one finds

$$n^s = \sum_{k|n} \sum_{d|k} \left(\frac{k}{d}\right)^s \mu(d).$$

This identity is obtained by Mobius inversion. Indeed, it is likely that this technique will handle any conjectured identity arising from a multiplicative function.

4. A q-series Example

If one expects a nice LU decomposition, the sequence of $n \times n$ determinants should also be nice. In Problem 10859 from the American Mathematical Monthly, March 2004, v.111 (3), pp.260–261, one finds a solution to the following problem:

> Compute the determinant of the $n \times n$ matrix whose (i, j)-entry is $q^{(i-j)^2}$ for $0 \leq i, j \leq n$.

The solution is given as

$$\prod_{k=1}^{n-1}(1-q^{2k})^{n-k}.$$

The form suggests trying an LU factorization. Maple produces nice polynomial entries. For example, one has

$$\begin{bmatrix} 1 & q & q^4 & q^9 & q^{16} \\ q & 1 & q & q^4 & q^9 \\ q^4 & q & 1 & q & q^4 \\ q^9 & q^4 & q & 1 & q \\ q^{16} & q^9 & q^4 & q & 1 \end{bmatrix} = LU$$

where

$$L = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ q & 1 & 0 & 0 & 0 \\ q^4 & \left(q^2+1\right)q & 1 & 0 & 0 \\ q^9 & \left(q^4+q^2+1\right)q^4 & \left(q^4+q^2+1\right)q & 1 & 0 \\ q^{16} & \left(q^6+q^4+q^2+1\right)q^9 & \left(q^8+q^6+2\,q^4+q^2+1\right)q^4 & \left(q^6+q^4+q^2+1\right)q & 1 \end{bmatrix}$$

and

$$U = \begin{bmatrix} 1 & q & q^4 & q^9 & q^{16} \\ 0 & 1-q^2 & q-q^5 & q^4-q^{10} & q^9-q^{17} \\ 0 & 0 & 1-q^4-q^2+q^6 & q-q^7-q^5+q^{11} & q^4-q^{12}-q^{10}+q^{18} \\ 0 & 0 & 0 & 1-q^{12}+q^{10}+q^8-q^4-q^2 & q+q^{15}+q^{13}-q^9-q^7-q^5+q^{11}-q^{19} \\ 0 & 0 & 0 & 0 & 1+q^{20}-q^{18}-q^{16}+2\,q^{10}-q^2-q^4 \end{bmatrix}$$

Factoring eventually produces the (i,j) term of L to be

$$L_{ij} = \begin{cases} q^{(i-j)^2}\prod_{k=1}^{j-1}\frac{1-q^{2i-2k}}{1-q^{2k}}, & i \geq j, \\ 0, & \text{otherwise}, \end{cases}$$

while the (i,j) term of U is

$$U_{ij} = \begin{cases} q^{(i-j)^2}\prod_{k=1}^{i-1}(1-q^{2j-2k}), & i \leq j, \\ 0, & \text{otherwise}. \end{cases}$$

These produce

$$q^{(i-j)^2} = \sum_{m=1}^{\min(i,j)}\left(q^{(i-m)^2+(j-m)^2}\prod_{k=1}^{m-1}\frac{\left(1-q^{2\,i-2\,k}\right)\left(1-q^{2\,j-2\,k}\right)}{1-q^{2\,k}}\right)$$

with the special case $i=j$ generating

$$1 = \sum_{m=1}^{i}\left(q^{2\,(i-m)^2}\prod_{k=1}^{m-1}\frac{\left(1-q^{2\,i-2\,k}\right)^2}{1-q^{2\,k}}\right).$$

This new identity may be written in terms of q-series. Computer aided proofs can be found with q-versions of the WZ algorithm; see [4].

5. Orthogonal Polynomials

Since orthogonal polynomials satisfy recurrence relationships, it is not surprising that they may also yield beautiful formulas using the LU approach.

The Chebyshev polynomials are the simplest class of orthogonal polynomials to study. Let $T_i(x)$ represent the i^{th} Chebyshev polynomial of the first kind. By factoring the matrix whose (i,j) entry is $T_{i+j-2}(x)$, we find

$$\begin{bmatrix} 1 & x & -1+2\,x^2 & 4\,x^3-3\,x \\ x & -1+2\,x^2 & 4\,x^3-3\,x & 1+8\,x^4-8\,x^2 \\ -1+2\,x^2 & 4\,x^3-3\,x & 1+8\,x^4-8\,x^2 & 16\,x^5-20\,x^3+5\,x \\ 4\,x^3-3\,x & 1+8\,x^4-8\,x^2 & 16\,x^5-20\,x^3+5\,x & -1+32\,x^6-48\,x^4+18\,x^2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ x & 1 & 0 & 0 \\ -1+2\,x^2 & 2\,x & 1 & 0 \\ 4\,x^3-3\,x & 4\,x^2-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & -1+2\,x^2 & 4\,x^3-3\,x \\ 0 & -1+x^2 & 2\,x^3-2\,x & 1+4\,x^4-5\,x^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This leads to the identity

$$T_{n+m} = T_n T_m + \frac{(T_{m+1}-xT_m)(T_{n+1}-xT_n)}{x^2-1}. \tag{5.1}$$

Similarly, let $U_i(x)$ represent the i^{th} Chebyshev polynomial of the second kind. By factoring the matrix whose (i,j) entry is $U_{i+j-2}(x)$, we find

$$\begin{bmatrix} 1 & 2\,x & 4\,x^2-1 & 8\,x^3-4\,x \\ 2\,x & 4\,x^2-1 & 8\,x^3-4\,x & 1+16\,x^4-12\,x^2 \\ 4\,x^2-1 & 8\,x^3-4\,x & 1+16\,x^4-12\,x^2 & 32\,x^5-32\,x^3+6\,x \\ 8\,x^3-4\,x & 1+16\,x^4-12\,x^2 & 32\,x^5-32\,x^3+6\,x & -1+64\,x^6-80\,x^4+24\,x^2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2\,x & 1 & 0 & 0 \\ 4\,x^2-1 & 2\,x & 1 & 0 \\ 8\,x^3-4\,x & 4\,x^2-1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2\,x & 4\,x^2-1 & 8\,x^3-4\,x \\ 0 & -1 & -2\,x & 1-4\,x^2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

This implies the identity

$$U_{m+n} = U_m U_n - U_{m-1} U_{n-1}. \tag{5.2}$$

Both equations 5.1 and 5.2 are easily proven using well-known properties of Chebyshev polynomials.

The Legendre polynomials are not as easy to work with. Let the (i,j) entry of a matrix be $P_{i+j-2}(x)$, where $P_k(x)$ is the k^{th} Legendre polynomial.

The challenge in understanding this LU decomposition manifests itself in that the derivative of one column is a scaled version of the succeeding column. This produces L with an (i, j) entry of

$$\frac{2^{j-1}(i-1)!}{(i+j-2)!}\frac{d^{j-1}P_{i-1}(x)}{dx^{j-1}}$$

and U with an (i, j) entry of

$$\frac{2^{2-i}(j-1)!}{(j+i-2)!}\frac{d^{i-1}P_{j-1}(x)}{dx^{i-1}}(x^2-1)^{i-1}$$

for $i \geq 2$ and $P_{j-1}(x)$ for $i = 1$. These combine to give the identity

$$P_{i+j}(x) = P_i(x)P_j(x)+2\sum_{m=1}^{\min\{i,j\}}\frac{i!j!}{(i+m)!(j+m)!}(x^2-1)^m\frac{d^m P_i(x)}{dx^m}\frac{d^m P_j(x)}{dx^m}$$

The special case $i = j$ delivers

$$P_{2i}(x) = P_i(x)^2 + 2\sum_{m=1}^{i}\left(\frac{i!}{(i+m)!}\frac{d^m P_i(x)}{dx^m}\right)^2(x^2-1)^m$$

Neither of these formulas could be proven and so stand as conjectures.

6. Miscelaneous

A problem from the American Mathematical Monthly (problem 10464 in the June-July issue of 1998) gives a nice closed form for the determinant of the $n \times n$ matrix whose (i, j) entry is $\partial^{i+j-2}e^{xy}/\partial x^{i-1}\partial y^{j-1}$. Using the LU approach, one finds L has an (i, j) entry of

$$\binom{i-1}{j-1}y^{i-j}$$

while the (i, j) entry of U is

$$\binom{j-1}{i-1}x^{j-i}(i-1)!.$$

This gives

$$\frac{\partial^{i+j}e^{xy}}{\partial x^i \partial y^j} = e^{xy}\sum_{k=0}^{\min\{i,j\}}\binom{i}{k}\binom{j}{k}k!x^{j-k}y^{i-k}$$

This formula may be proven using induction.

Another American Mathematical Monthly problem (# 10387 in the April issue of 1997) looks at the determinant of a Hankel matrix where the (i, j) term is $\tan(i + j - 1)x$. While there is a closed form, a much more general result is found: the determinant of the matrix whose (i, j) entry is

$$\frac{ax_i + by_j}{x_i + y_j}$$

is

$$\frac{(a-b)^{n-1}(a\prod x_i+(-1)^{n-1}b\prod y_j)\prod_{1\leq i\leq j\leq n}((x_i-x_j)(y_i-y_j))}{\prod_{i,j=1}^{n}(x_i+y_j)}.$$

Investigating this further, one follows the LU approach. While a proof could not be produced, it is conjectured that the L matrix has an (i,j) term of

$$L_{ij}=\frac{\prod_{k=1}^{j}(x_j+y_k)\prod_{k=1}^{j-1}(x_k-x_i)\left(ax_i\prod_{k=1}^{j-1}x_k+(-1)^{j-1}\,b\prod_{k=1}^{j}y_k\right)}{\prod_{k=1}^{j-1}(x_k-x_j)\prod_{k=1}^{j}(x_i+y_k)\left(a\prod_{k=1}^{j}x_k+(-1)^{j-1}\,b\prod_{k=1}^{j}y_k\right)}$$

while the U matrix has an (i,j) term of

$$U_{ij}=\frac{(a-b)\prod_{k=1}^{i-1}(x_k-x_i)\prod_{k=1}^{i-1}(y_k-y_j)\left(a\prod_{k=1}^{i}x_k+(-1)^{i-1}\,by_j\prod_{k=1}^{i-1}y_k\right)}{\prod_{k=1}^{i-1}(x_i+y_k)\prod_{k=1}^{i}(y_j+x_k)\left(a\prod_{k=1}^{i-1}x_k+(-1)^{i}\,b\prod_{k=1}^{i-1}y_k\right)}.$$

The associated identity could not be proven and so stands an unsolved conjecture.

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