THE MEAN-MEDIAN MAP

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1. Introduction

In the last few decades mathematicians have discovered very simple functions which induce spectacularly intricate dynamics. For example, the logistic map produces a striking bifurcation diagram and fractals, the Hénon and Lorentz maps display complex strange attractors, and the Game of Life for cellular automata has the power of a universal Turing machine. These iteration schemes, and others not explicitely mentioned here, have grabbed public attention because of the beautiful patterns associated with them. The book [1] revels in displaying their rich dynamics.

In [2], Schultz and Shiflett propose a new map on *sets* with dynamics that are both complex and ordered. Given three real numbers a, b, c they define x_4 as the solution of the equation

$$\frac{a+b+c+x_4}{4} = median(a,b,c). \tag{1.1}$$

By iterating the scheme suggested by (1.1) one obtains the recursive equality

$$\frac{a+b+c+x_4+\cdots+x_n}{n} = median(a,b,c,x_4,\ldots,x_{n-1}). \tag{1.2}$$

Schultz and Shifflet [2] discuss the complicated dynamics associated to (1.2) and conjecture that the sequence $\{x_n\}_{n=4}^{\infty}$ becomes constant after finitely many iterations.

Building on their work, we will introduce a new function — the mean-median map — which will also have a rich structure and whose graph (Figure 3) is a new icon of rich dynamics stemming from simple rules. While we are not able to solve the conjecture of Schultz and Shiflett, we are adding to their analysis some results, strategies and conjectures that may inspire other mathematicians to further investigate the problem and hopefully find a definite answer to all questions raised in [2] and by us.

This paper is organized in the manner we now describe. Section 2 contains results and conjectures about the problem in the general case when the number of starting values is not necessarily 3. In Section 3 we analyze the case studied in [2] and add some new results to the ones found by Schultz and

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Shiflett. For comparison, we finish with a figure to illustrate the complexities when 4 initial points are taken.

2. Results and Conjectures

Starting with a non-empty finite set $S_n = \{x_1, \dots, x_n\} \subset \mathbb{R}$, generate the unique number x_{n+1} which satisfies the *mean-median equation*

$$\frac{x_1 + \cdots + x_n + x_{n+1}}{n+1} = median(S_n). \tag{2.1}$$

As usual, we define the median of the set $S_n = \{x_1, \dots, x_n\}$, where $x_1 \leq \cdots \leq x_n$, as

$$median(S_n) = \begin{cases} x_{(n+1)/2}, & \text{n odd,} \\ \frac{x_{n/2}+x_{n/2+1}}{2}, & \text{n even.} \end{cases} \tag{2.2}$$

The set S_n is augmented to $S_{n+1} := S_n \cup \{x_{n+1}\}$. By applying the mean-median equation repeatedly to a set S_n one generates an infinite sequence $\{x_k\}_{k=n+1}^{\infty}$. Our goal is to investigate the behavior of such sequences. The first important result was also observed in [2] for the case $n = 3$.

Theorem 2.1. *The sequence of medians is monotone.*

Proof. Starting with $S_n = \{x_1, \dots, x_n\} \subset \mathbb{R}$, we have

$$x_1 + \cdots + x_n + x_{n+1} = (n+1)\ median(S_n).$$

The next iteration produces

$$x_1 + \cdots + x_n + x_{n+1} + x_{n+2} = (n+2)\ median(S_{n+1}).$$

Subtracting yields

$$x_{n+2} = (n+1)[median(S_{n+1}) - median(S_n)] + median(S_{n+1}). \tag{2.3}$$

If $median(S_{n+1}) \geq median(S_n)$, then $x_{n+2} \geq median(S_{n+1})$, which in turn forces $median(S_{n+2}) \geq median(S_{n+1})$ by the definition of the median. This process may be continued indefinitely producing a sequence of medians which is monotonically non-decreasing. Similarly, if $median(S_{n+1}) \leq median(S_n)$, the sequence of medians is monotonically non-increasing. □

It is easy to check that when the starting set has one or two elements the sequence of medians settles immediately. However, when the starting set has three points $a < b < c$, the sequence reveals surprisingly complicated dynamics. As mentioned in the introduction, the main conjecture of [2] is that the sequence of medians is not only monotonic, but eventually fixed. We reformulate the conjecture for a set S_n with $n \geq 3$ and for the sequence $\{x_k\}_{k=n+1}^{\infty}$.

Strong Terminating Conjecture: For every finite non-empty set $S \subset \mathbb{R}$, there exists an integer k such that the associated infinite sequence satifies $x_j = x_k$ for all $j > k$. In other words, the sequence of new terms settles permanently to the median after a finite number of mean-median iterations.

With the help of Maple, we studied the behavior of $\{x_k\}_{k=n+1}^{\infty}$ for many cases. These numerical investigations suggest that the number of steps needed until the limiting median is attained is an unbounded function. This leads to the

Weak Terminating Conjecture: For every finite non-empty set $S \subset \mathbb{R}$, the limit of the medians is finite.

To reduce the iteration scheme to a canonical collection of starting sets, we start with the following linearity result. Let L denote the map which takes a starting set $S_n = \{x_1,\ x_2, \dots,\ x_n\}$ and produces the set $L(S_n) = \{y_1,\ y_2, \dots,\ y_n\}$, $y_i = ax_i + b,\ i = 1,\ 2,\ \dots,\ n$ and $a \neq 0$.

Theorem 2.2. *The sequence $\{y_n\}_{n=1}^{\infty}$ is convergent if and only if the sequence $\{x_n\}_{n=1}^{\infty}$ is convergent.*

Proof. We shall prove that if the sequence $\{x_n\}_{n=1}^{\infty}$ converges to x_∞ then the sequence $\{y_n\}_{n=1}^{\infty}$ converges to $ax_\infty + b$. The result is a consequence of (2.3) and the equality

$$\sum_{i=1}^{i=n} \frac{ax_i + b}{n} = a \sum_{i=1}^{i=n} \frac{x_i}{n} + b.$$

□

Let $M : \mathbb{R}^n \to \mathbb{R}$ be the function that assigns to every n-tuple $(x_1, x_2, \dots,\ x_n)$ in $\mathbb{R}^n$ the limit x_∞ of the sequence $\{x_k\}_{k=n+1}^{\infty}$ defined by the *mean-median* equation (2.1), provided that such a limit exists. Theorem 2.2 and numerical evidence suggest the following conjecture.

Continuity Conjecture
The function M is continuous.

We would like to mention two results that may make it easier to provide an answer to the stated conjectures. The first result can be easily derived from (2.3).

Theorem 2.3. *The sequence of the medians converges in finitely many steps if and only if the sequence $\{x_k\}_{k=n+1}^{\infty}$ becomes stationary.*

Theorem 2.4. *Assume that $x_i = x_j$ for some $j > i > n$. Then the sequence of the medians converges.*

Proof. We can assume, without loss of generality, that the sequence of the medians is monotone non-decreasing. From (2.3) we derive that $x_i \geq median(S_{i-1})$ and $x_j \geq median(S_{j-1})$. If the sequence of the medians does not reach the value $x_i = x_j$ then it is convergent. In the case when the sequence of the medians reaches the value $x_j = x_i$, two successive medians will be equal. This result implies that $\{x_k\}_{k=n+1}^{\infty}$ becomes stationary. □

We point out that every numerical investigation containes the presence of two equal values $x_j = x_i$ before the sequence becomes stationary. We have never found an example of convergence without this feature. We cannot explain the occurrence of the two equal values, which may be separated by a large number of iterations.

3. Three Initial Points

Lemma 2.2 implies that the dynamics of the set $S = \{a, b, c\}$, with $a \leq b \leq c$, are equivalent to those of $\{0, x, 1\}$ where $x = (b-a)/(c-a) \in [0, 1]$. Shultz and Schiflett consider the reduction of all cases to $\{0, z, z+1\}$ instead. Let $M_k(\{0, x, 1\})$ represent the median after k elements are in the set and $m(x) = M_\infty(\{0, x, 1\})$. Examples are $m(1/2) = 1/2$, $m(2/3) = 1$, $m(1) = 1$. The following result with $L(z) = 1 - z$ from Theorem 2.2 implies that one need only work on the interval $[1/2, 1]$.

Theorem 3.1. *For all* $0 \leq a \leq 1$, $m(1-a) = 1 - m(a)$.

The following is a less obvious use of Theorem 2.2.

Theorem 3.2. *If* $1/2 \leq a \leq 1$, *then*

$$m(a) = (3a-1)m\left(\frac{a}{3a-1}\right) \tag{3.1}$$

Proof. If $2/3 \leq a \leq 1$, then

$$\begin{aligned} m(a) &= M_\infty(\{0, a, 1\}) \\ &= M_\infty(\{0, a, 1, 3a-1\}) \\ &= M_\infty(\{0, a, 3a-1\}) \\ &= (3a-1)M_\infty(\{0, a/(3a-1), 1\}) \\ &= (3a-1)m\left(\frac{a}{3a-1}\right) \end{aligned}$$

The equation is invariant on replacing a with $a/(3a-1)$ which settles the case when $1/2 \leq a \leq 2/3$. □

This result implies that one need only work on the interval $[1/2, 2/3]$. Theorem 3.2 cannot be used in a recursive manner because replacing a with $a/(3a-1)$ leaves Equation 3.1 unchanged. Given some $a \in [1/2, 1]$, we refer to $a/(3a-1)$ as its *partner*.

The behavior of m near $a = 1/2$ (and hence $a = 1$) is completely determined. The following is equivalent to the strongest result in [2].

Theorem 3.3. *For all* $\epsilon \in [0, 4/333]$,

$$m\left(\frac{1}{2} + \epsilon\right) = \frac{1}{2} + \frac{333}{8}\epsilon \tag{3.2}$$

Proof. We wish to study $M_k(\{0, 1/2+\epsilon, 1\})$. For sufficiently small ϵ, these new terms stay less than one and take the form $1/2 + r\epsilon$ for some rational number r. In chronological order, the corresponding r terms are:
3, 6, 8, 13.5, 16.5, 15, 17, 38.25, 43.75, 23.25, 24.75, 26.25, 27.75, 20.75, 21.25, 52.625, 56.375, 26.5, 44.25, 46.25, 42, 43.5, 41.625, 42.875, 29.625, 29.875, 30.125, 30.375, 47.125, 48.375, 60.5625, 62.4375, 34.25, 34.5, 34.75, 35, 35.25, 35.5, 113.6875, 117.5625, 39.875, 40.125, 40.375, 40.625, 40.875, 41.125, 41.375, 41.625, 41.875, 42.125, 111.125, 113.875, 84.5625, 86.1875, 47.25, 47.5, 47.75, 48, 48.25, 48.5, 48.75, 49, 49.25, 49.5, 49.75, 50, 50.25, 50.5, 41.625
The last term repeats indefinitely. The largest term, $1/2 + 117.5625\epsilon$, will be less than 1 if $\epsilon < 0.004253$. Since the median settles to $1/2 + 41.625\epsilon = 1/2 + 333\epsilon/8$, we have the desired equation. Note that $r = 41.625$ was attained two times before this point. The upper bound of $\epsilon < 0.004253$ can be increased and still yield the same dynamics. As long as the *median* stays less than 1, this will occur. Since the median is increasing and the final median is $1/2 + 41.625\epsilon$, we can allow $\epsilon < 4/333 \approx 0.012012$. For all such ϵ, the median is permanently attained after 70 iterations. □

Theorem 2.1 implies that the sequence of medians is non-decreasing for each $a \in [1/2, 1]$, hence the preceeding proof implies

Corollary 3.1. For $a \in [1/2 + 4/333, 1]$, $m(a) \geq 1$.

We can see how the conjectured function m is built by looking at the median of $\{0, x, 1\}$ after n steps; see Figure 1. By Theorem 2.1, the iterates form a sequence of non-decreasing piecewise linear functions. Several results follow from this. If the Weak Terminating Conjecture holds and m is continuous, Dini's Theorem claims that the convergence is uniform.

Regarding the corners of m, we have the

Theorem 3.4. *If $x = x_0$ is a corner of m, then x_0 is rational.*

Proof. Because the mean and median of a set of rationals is always rational, if the corners of $M_k(\{0, x, 1\})$ are at rational points, the corners of $M_{k+1}(\{0, x, 1\})$ must also be at rational points. Since $M_3(\{0, x, 1\}) = x$, induction implies the function $M_k(\{0, x, 1\})$ must also have all of its corners at rational numbers for each $k \geq 4$. If m has a corner at x_0, then M_k converges after finitely many iterations, and hence x_0 must be a rational. □

With Corollary 3.1 in mind, a simple, better lower bound for m does not exist since $m = 1$ at the following values:

$$\frac{2}{3}, \frac{7}{12} \frac{8}{15}, \frac{9}{16}, \frac{9}{17}, \frac{18}{35}, \frac{27}{52}, \frac{55}{106}, \frac{71}{138}, \frac{73}{142}, \frac{141}{274}, \frac{143}{278}, \frac{145}{282}, \frac{157}{306}, \frac{327}{638}, \frac{329}{642},$$
$$\frac{331}{646}, \frac{333}{650}, \frac{335}{654}, \frac{337}{658}, \frac{339}{662}, \frac{341}{666}.$$

Note that all of these fractions have the form

$$\frac{1}{a}\left\lceil \frac{341a}{666} \right\rceil$$

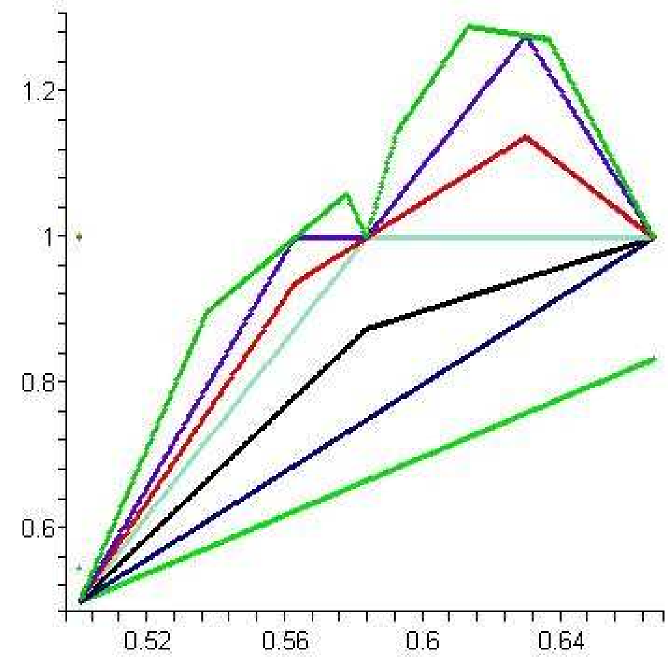

FIGURE 1. The medians $M_k(\{0, x, 1\})$, $k = 4, \ldots, 10$.

for some integer a. Corollary 3.1 implies that checking the corners of M_{70} generates all the finitely-many values of x where $m(x) = 1$.

The reason the dynamics are understood on the interval $[1/2, 1/2+4/333]$ is that the ordering of the points does not change. To the right of this interval, however, the ordering changes. Numerically, the first bifurcation occurs at $x \approx 0.51208253563071686875132$ and the median is permanently attained after 72 iterations. The following bifurcation occurs at $x \approx 0.51208259196424797177794$ and the median is permanently attained after 74 iterations. After this the median is permanently attained after 76 iterations. In each of these parameter ranges, however, we still have $m(1/2 + \epsilon) = 1/2 + 41.625\epsilon$.

One would hope that Theorem 3.1 could be exploited at $a = 2/3$. For example, if m is differentiable at $2/3$, then that derivative equals zero. However, numerical evidence suggests that m is affine on each side of this point: for sufficiently small $\epsilon > 0$,

$$m\left(\frac{2}{3} + \epsilon\right) = 1 + \frac{231}{2}\epsilon,\ m\left(\frac{2}{3} - \epsilon\right) = 1 + \frac{225}{2}\epsilon$$

Combining this observation with Theorem 3.3 and particularly with Figure 1 suggests the

Affine Segments Conjecture: The function m is affine off of a set of measure zero.

There is reason to hope that the map m is affine almost everywhere. Specifically, numerical evidence suggests that m is differentiable almost everywhere and on any interval where it is differentiable, the derivative is constant. An easily-proven related observation is that if m is twice continuously differentiable, then equation (3.1) implies

$$(3a-1)^3 m''(a) = m''\left(\frac{a}{3a-1}\right),$$

so if m is affine on an interval, it is also affine on its partner interval.

A coarse calculation gives a sketch of the function m over a short interval; see Figure 2. If nothing else, the figure shows that the map m is very complicated. As an indication of the sensitivity of m, note that $m(0.6) = 2.4$ with 32 steps, while $m(0.60001) = 2.635316642924070358276367187 5$ with 12488 steps. The largest value found so far is $m(0.843) = 4.64546875$, in 526 steps.

The case with four initial points has some similarities as with three initial points. Theorem 2.2 allows us to consider only the initial points $0 \leq a \leq b \leq 1$. The function $m = m(a, b)$ on the unit square is displayed in figure 3.

References

[1] Peitgen, h.-O., Jürgens, H. and Saupe, D. *Chaos and Fractals*. Springer, New York, 1992.

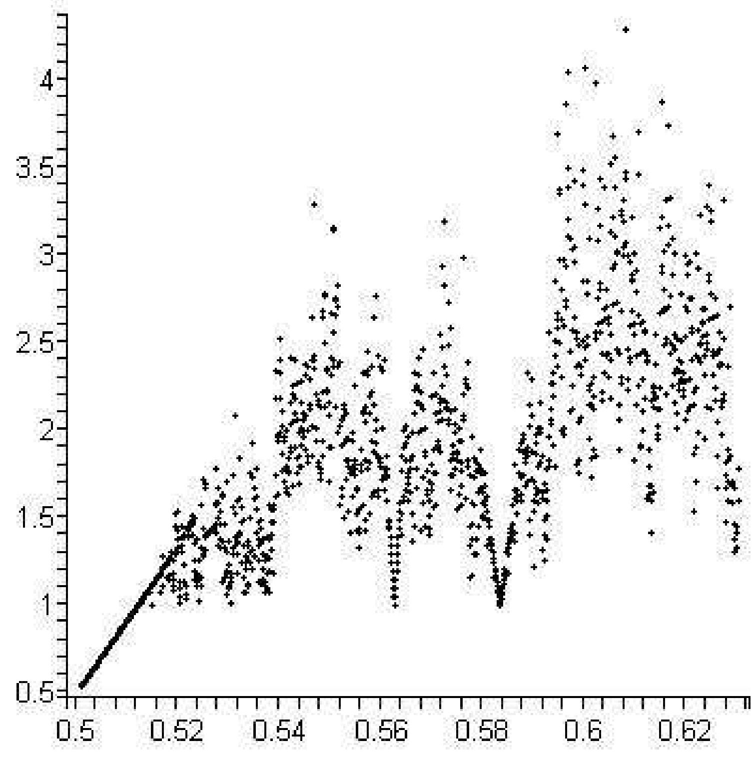

FIGURE 2. The function m on $[1/2, 2/3]$.

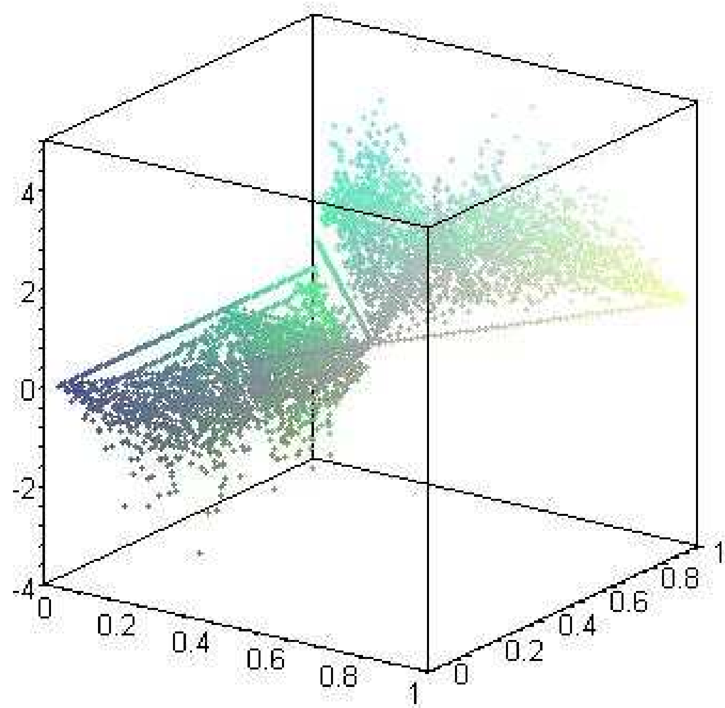

FIGURE 3. The function m with four initial points.

[2] Schultz, H. and Shiflett, R. M&m Sequences. The College Mathematics Journal, 36(2005), 191–198.

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