DYNAMICS OF THE DEGREE SIX LANDEN TRANSFORMATION

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ABSTRACT. We establish the basin of attraction for the fixed point $(3, 3)$ of a dynamical system arising from the evaluation of a definite integral.

1. INTRODUCTION

The transformation theory of elliptic integrals was initiated by Landen in [6, 7], wherein he proved the invariance of the function

$$G(a, b) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \tag{1.1}$$

under the transformation

$$a_1 = (a + b)/2 \qquad b_1 = \sqrt{ab} \tag{1.2}$$

i.e. that

$$G(a_1, b_1) = G(a, b). \tag{1.3}$$

Gauss [4] rediscovered the invariance of $G(a, b)$ under the transformation (1.2) while numerically calculating the length of a lemniscate. The Gauss-Landen transformation can be iterated to produce a convergent double sequence (a_n, b_n) that satisfies $0 < a_n - b_n < 2^{-n}$. The common limit is the famous *arithmetic-geometric mean* of a and b denoted by $AGM(a, b)$. Passing to the limit in $G(a, b) = G(a_n, b_n)$ yields

$$\frac{\pi}{2\, AGM(a, b)} = \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}}. \tag{1.4}$$

Thus the elliptic integral $G(a, b)$ can be evaluated as a limit of a recursively defined sequence. Information about these topics can be obtained in [3].

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The existence of a Landen transformation for the integral

$$U_6(a,b;c,d,e) = \int_0^\infty \frac{cx^4+dx^2+e}{x^6+ax^4+bx^2+1}\,dx \tag{1.5}$$

was established in [1]. Indeed the integral U_6 is invariant under the transformation

$$\begin{aligned} a_{n+1} &= \frac{a_nb_n+5a_n+5b_n+9}{(a_n+b_n+2)^{4/3}} \\ b_{n+1} &= \frac{a_n+b_n+6}{(a_n+b_n+2)^{2/3}} \\ c_{n+1} &= \frac{c_n+d_n+e_n}{(a_n+b_n+2)^{2/3}} \\ d_{n+1} &= \frac{(b_n+3)c_n+2d_n+(a_n+3)e_n}{a_n+b_n+2} \\ e_{n+1} &= \frac{c_n+e_n}{(a_n+b_n+2)^{1/3}}. \end{aligned} \tag{1.6}$$

The first two equations in (1.6) are independent of the variables c, d and e so they define a map

$$\Phi_6(a,b) = \left(\frac{ab+5a+5b+9}{(a+b+2)^{4/3}}, \frac{a+b+6}{(a+b+2)^{2/3}} \right) \tag{1.7}$$

that is well-defined on $\mathbb{R}^2$ minus the line $a+b+2=0$. The goal of this paper is to provide a purely dynamical proof of

Main Theorem: *The basin of attraction for the fixed point* $(3,3)$ *of the dynamical system*

$$\begin{aligned} a_{n+1} &= \frac{a_nb_n+5a_n+5b_n+9}{(a_n+b_n+2)^{4/3}} \\ b_{n+1} &= \frac{a_n+b_n+6}{(a_n+b_n+2)^{2/3}} \end{aligned} \tag{1.8}$$

is the region of the (a,b)*-plane for which the integral* (1.5) *converges.*

Section 2 describes the origin of these transformations: they appear from a sequence of elementary changes of variables that preserve a rational integral. In section 3 we prove that the sequence (a_n,b_n) converges to $(3,3)$ provided the initial point is on the first quadrant. Section 4 contains a description of the region Λ on the (a,b)-plane on which the integral U_6 converges. This is given in terms of the **discriminant curve** $\mathfrak{R}$ defined by

$$R(a,b) := 4a^3+4b^3-18ab-a^2b^2+27=0. \tag{1.9}$$

Section 5 describes a relation between the curve $\mathfrak{R}$ and the range of the map Φ_6 that defines (1.8). This gives a rational parametrization of $\mathfrak{R}$ that is used to describe the dynamics of Φ_6 on it. Section 6 presents the fixed points of

Φ_6 and their linearizations. Finally Section 7 presents the proof of the main result.

Then the set $R(a,b) = 0$ is a real algebraic curve with two connected components $\mathfrak{R}_{\pm}$. The component $\mathfrak{R}_{+}$, of equation $R_{+}(a,b) = 0$ is contained in the first quadrant and contains the point $(3,3)$ as a cusp. The second component $\mathfrak{R}_{-}$ given by $R_{-}(a,b) = 0$, is disjoint from the first quadrant.

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FIGURE 1. The discriminant curve

The identity

$$R(a_1, b_1) = \frac{(a-b)^2 R(a,b)}{(a+b+2)^4}, \tag{1.10}$$

plays an important role in the dynamics of the (1.8). The proof is elementary. In particular it follows from here that the discriminant locus $R(a,b) = 0$, and the regions $\{(a,b) : R(a,b) > 0\}$, located between the two branches in figure 1, and $\{(a,b) : R(a,b) < 0\}$ are preserved by Φ_6.

The identity (1.10) also shows that the diagonal $\Delta = \{(a,b) : a = b\}$ of $\mathbb{R}^2$ is mapped onto the discriminant curve $\mathfrak{R}$. This yields the parametrization

$$a(t) = \frac{t+9}{2^{4/3}(t+1)^{1/3}}$$

$$b(t) = \frac{2^{1/3}(t+3)}{(t+1)^{2/3}}$$

of this curve. This parametrization will be obtained in Section 5 as a consequence of the analysis of mapping properties of Φ_6.

The elementary result of Section 2 states that

Theorem 1.1. The boundary of the convergent set Λ is the curve $\mathfrak{R}_{-}$. Thus

$$U_6(a,b;c,d,e) = \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1}\,dx \tag{1.11}$$

is finite if and only if $R_{-}(a,b) > 0$.

The method described in Section 2 was used in [2] to produce a Landen transformation for the integral of any even rational function, that is, given an even rational function R there is a new one R_{+} such that

$$\int_0^\infty R(x)\,dx = \int_0^\infty R_{+}(x)\,dx. \tag{1.12}$$

For example, the integral

$$U_8(a,b,c;d,e,f,g) \quad := \quad \int_0^\infty \frac{dx^6+ex^4+fx^2+g}{x^8+ax^6+bx^4+cx^2+1}\,dx \tag{1.13}$$

is invariant under the transformation

(1.14)

$$\begin{aligned}
a_{n+1} &= \frac{b_n(a_n+c_n)+4a_nc_n+10(a_n+c_n)+8(b_n+2)}{(a_n+b_n+c_n+2)^{3/2}} \\
b_{n+1} &= \frac{a_nc_n+6(a_n+c_n)+2(b_n+10)}{a_n+b_n+c_n+2} \\
c_{n+1} &= \frac{a_n+c_n+8}{(a_n+b_n+c_n+2)^{1/2}} \\
d_{n+1} &= \frac{d_n+e_n+f_n+g_n}{(a_n+b_n+c_n+2)^{3/4}} \\
e_{n+1} &= \frac{g_n(3a_n+b_n+6)+f_n(a_n+4)+e_n(c_n+4)+d_n(3c_n+b_n+6)}{(a_n+b_n+c_n+2)^{5/4}} \\
f_{n+1} &= \frac{g_n(a_n+5)+f_n+e_n+d_n(c_n+5)}{(a_n+b_n+c_n+2)^{3/4}} \\
g_{n+1} &= \frac{g_n+d_n}{(a_n+b_n+c_n+2)^{1/4}}.
\end{aligned}$$

The convergence of this procedure has been established in [5] by showing that $R_+(x)dx$ is the direct image of the 1-form $\varphi = R(x)dx$ under the map

$$w = \pi(z) \quad = \quad \frac{z^2-1}{2z}. \tag{1.15}$$

In terms of the sections $\sigma_\pm(w) = w \pm \sqrt{w^2+1}$ the new form is

$$\pi_*\varphi \quad = \quad R(\sigma_+(w))\frac{d\sigma_+}{dw} + R(\sigma_-(w))\frac{d\sigma_-}{dw}. \tag{1.16}$$

The main result of [5] is that the dynamical system linked to these transformations converges precisely on the region where the original integral is finite. In this paper we provide a purely dynamical proof of this result, in the case of degree 6.

2. The transformation for U_6

A polynomial $P_d(x)$ of degree d is called *symmetric* if $P_d(1/x) = x^{-d}P_d(x)$. A symmetric polynomial $P_d(x)$ is said to be *normalized* if it is monic. For example the normalized polynomial of degree 6 is $P_6(x) = x^6+ax^4+ax^2+1$ and that of degree 12 is

$$P_{12}(x) = (x^{12}+1) + a_3(x^{10}+x^2) + a_2(x^8+x^4) + 2a_1x^6.$$

The first step in the derivation of (1.6) is to symmetrize the denominator of the integrand, producing an integral in which the degree of the denominator is double that of the original. We then employ a sequence of elementary substitutions to transform the new integral back to one with the original degree.

Proposition 2.1. Let $R_4(x) = cx^4 + dx^2 + e,\ Q_6(x) = x^6 + ax^4 + bx^2 + 1$, $R_{10}(x) = R_4(x)(x^6 + bx^4 + ax^2 + 1)$ and $P_{12}(x)$ be the normalized polynomial of degree 12. Then

$$\int_0^\infty \frac{R_4(x)}{Q_6(x)}dx \quad = \quad \int_0^\infty \frac{R_{10}(x)}{P_{12}(x)}dx. \tag{2.1}$$

Proof. Observe that $P_{12}(x) = x^6 Q_6(x) Q_6(1/x)$ and $R_{10}(x) = x^6 R_4(x) Q_6(1/x)$. □

Now transform the integral in (2.1) using the change of variables $x = \tan\theta$ to produce

$$U_6 \quad = \quad \int_0^{\pi/2} \left(\sum_{k=0}^{5} r_k \cos^k 2\theta \right) \times \left(\sum_{k=0}^{3} s_{2k} \cos^{2k} 2\theta \right)^{-1} 2d\theta,$$

where $r_0, \cdots, r_5$ and $s_0, \cdots, s_6$ are functions of the parameters $a, \cdots, e$. For example, $r_0 = 2c + ac + bc + 2d + ad + bd + 2e + ae + be$, with similar expressions for the rest of them. Observe that the denominator is an even function of $\cos 2\theta$, so the terms with odd powers in the numerator have vanishing integral. Therefore, with $\psi = 2\theta$, we have

$$U_6 \quad = \quad 2\int_0^{\pi/2} \frac{r_4\cos^4\psi + r_2\cos^2\psi + r_0}{s_6\cos^6\psi + s_4\cos^4\psi + s_2\cos^2\psi + s_0}d\psi.$$

Letting $\theta = 2\psi$, we obtain

$$U_6 \quad = \quad \int_0^{\pi} \frac{t_2\cos^2\theta + t_1\cos\theta + t_0}{u_3\cos^3\theta + u_2\cos^2\theta + u_1\cos\theta + u_0}d\theta,$$

where $t_2, \cdots, t_0$ and $u_3, \cdots, u_0$ are again functions of the original parameters in U_6. Finally, the change of variables $y = \tan(\theta/2)$ yields

$$U_6 \quad = \quad \int_0^\infty \frac{v_4y^4 + v_2y^2 + v_0}{w_6y^6 + w_4y^4 + w_2y^2 + w_0}dy,$$

with $v_4, \cdots, v_0$ and $w_6, \cdots, w_0$ dependent upon a, b, c, d and e. The last step in the proof of (1.6) is to factor out w_0 and scale y to produce a monic polynomial in the denominator of the integrand.

3. Convergence for positive initial data

The first two equations of (1.6):

$$
\begin{aligned}
a_{n+1} &= \frac{a_nb_n+5a_n+5b_n+9}{(a_n+b_n+2)^{4/3}} \\
b_{n+1} &= \frac{a_n+b_n+6}{(a_n+b_n+2)^{2/3}}
\end{aligned}
\tag{3.1}
$$

are independent of c, d and e so they may be considered as a dynamical system on $\mathbb{R}^2$. In this section we prove that $(a_n, b_n) \to (3,3)$ provided $a_0, b_0 \geq 0$. The proof is a slight improvement over the one discussed in [1].

Theorem 3.1. Let $a_0 \geq 0$ and $b_0 \geq 0$. Then the sequence (a_n, b_n) defined in (1.8) converges to $(3,3)$.

Proof. It suffices to prove that

$$
(a_1-3)^2+(b_1-3)^2 \quad \leq \quad \frac{1}{2}\left[(a_0-3)^2+(b_0-3)^2\right], \tag{3.2}
$$

since iterating this inequality produces

$$
\left[(a_n-3)^2+(b_n-3)^2\right] \leq 2^{-n}\left[(a_0-3)^2+(b_0-3)^2\right]
$$

and we then have geometric convergence to $(3,3)$.

The inequality (3.2) is equivalent to

$$
\begin{aligned}
f(a,b) &= (a+b+2)^{8/3}(a^2+b^2-6a-6b-18)+2(a+b+2)^{4/3}(4ab+18a+18b+18-a^2-b^2)+ \\
&\quad +2(6a^3+6b^3+8a^2b+8ab^2+35a^2+35b^2-a^2b^2+78a+78b+52ab+63) \geq 0,
\end{aligned}
$$

and we need to prove that $f(a,b)$ has an absolute minimum of 0 at $(3,3)$. Note that $f(a,b) = f(b,a)$, so we may restrict the analysis to the region

$$
\Omega \quad = \quad \{(a,b) \in \mathbb{R}^2_+ : a \geq b\}. \tag{3.3}
$$

Introduce the new variables $x = (a+b+2)^{1/3}$ and $y = ab$, and write $h(x,y)$ for $f(a,b)$. The region Ω is then transformed into

$$
\Omega^* \quad = \quad \{(x,y) \in \mathbb{R}^2_+ : x \geq \sqrt[3]{2} \text{ and } 0 \leq y \leq \left(1-x^3/2\right)^2\},
$$

and in terms of the new variables, we need to prove that

$$
\begin{aligned}
h(x,y) \quad &= \quad x^{14}-10x^{11}-2x^{10}+12x^9-2x^8(y+1)+44x^7- \\
&\quad -2x^6+4x^4(3y-11)-20x^3(y-1)-2(y-1)^2 \geq 0
\end{aligned}
$$

for $(x,y) \in \Omega^*$. This will be achieved by showing that $h \geq 0$ on the upper part of the boundary of Ω^* and that $h_y(x,y) < 0$.

First observe that

$$
\begin{aligned}
h_y(x,y) \quad &= \quad -2(2y+x^8-6x^4+10x^3-2) \\
&= \quad -4y-2P(x-1)
\end{aligned}
$$

with

$$P(x) \quad = \quad x^8 + 8x^7 + 28x^6 + 56x^5 + 64x^4 + 42x^3 + 22x^2 + 14x + 3.$$

Thus $h_y(x, y) < 0$ for $(x, y) \in \Omega^*$.

Now we examine the values of h on the upper part of the boundary of Ω^*. Along the curve $y = (1 - x^3/2)^2$, $x \geq \sqrt[3]{2}$, we have

$$\begin{aligned} h(x,(1-x^3/2)^2) \quad &= \quad \tfrac{1}{8}x^4(x-2)^2 \times (4(x-1)^8 + 48(x-1)^7 + 271(x-1)^6 + 902(x-1)^5 + \\ &\quad + 1905(x-1)^4 + 2628(x-1)^3 + 2289(x-1)^2 + 1062(x-1) + 107), \end{aligned}$$

which has an absolute minimum of 0 at $x = 2$. The proof of the Theorem is complete. $\square$

Note. We have also shown in [1] that the convergence $(a_n, b_n) \to (3, 3)$ and the invariance of the integral U_6 under the transformation (1.6), yield the existence of a number L such that $(c_n, d_n, e_n) \to (1, 2, 1)L$. Passing to the limit in the invariance relation

$$\int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx \quad = \quad \int_0^\infty \frac{c_n x^4 + d_n x^2 + e_n}{x^6 + a_n x^4 + b_n x^2 + 1} dx$$

yields

$$\int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx \quad = \quad \frac{\pi}{2} \times L,$$

so that L is essentially the original integral. Thus we obtain an iterative scheme (by keeping track of c_n) to evaluate the integral of a rational function. The method converges quadratically.

4. The region of convergence for the integral U_6

We now describe the region

$$(4.1) \quad \Lambda \quad := \quad \left\{ (a, b) \in \mathbb{R}^2 : U_6 := \int_0^\infty \frac{cx^4 + dx^2 + e}{x^6 + ax^4 + bx^2 + 1} dx < \infty \right\},$$

where U_6 converges. Naturally this convergence depends only upon the roots of the denominator, so Λ is independent of the parameters in the numerator. We now provide an elementary characterization of the region Λ.

Step 1: The integral U_6 converges if and only if the polynomial $P(t) := t^3 + at^2 + bt + 1$ has no positive real roots. This follows from the partial fraction decomposition of the integrand.

Step 2: Suppose $a^2 \leq 3b$. Then $(a, b) \in \Lambda$. Observe that if $P(0) = 1$, the condition $a^2 \leq 3b$ implies that P is increasing so it does not have roots in $\mathbb{R}^+$.

From now on we assume $a^2 > 3b$.

Step 3: Suppose $a, b > 0$. Then $(a, b) \in \Lambda$. Let $t_+ = \frac{1}{3}(-a + \sqrt{a^2 - 3b})$ be the largest critical point of P. The condition $t_+ < 0$ is equivalent to $a,\, b > 0$. The conclusion follows from the fact that P is increasing for $t > t_+$ and $P(0) = 1$.

From now on we assume $t_+ > 0$.

Step 4: Suppose $a^2 > 3b$ and $t_+ > 0$. Then the condition $(a, b) \in \Lambda$ is equivalent to $P(t_+) > 0$. By Step 3 we only need to consider the case $t_+ > 0$. The equivalence of the two conditions is clear from the graph of P for $t > 0$. The condition $P(t_+) > 0$ reads

$$27 + 2a^3 - 9ab \quad > \quad 2(a^2 - 3b)^{3/2}. \tag{4.2}$$

Therefore, if $27 + 2a^3 - 9ab < 0$ then $(a, b) \notin \Lambda$. In the other case, squaring (4.2) yields $R_-(a, b) > 0$.

The previous steps give the convergence condition stated in Theorem 1.1.

5. The range of Φ_6

The study of mapping properties of Φ_6 is facilitated by studying the image of the line

$$L_c \quad := \quad \{(a, b) \in \mathbb{R}^2 : a + b = c\}. \tag{5.1}$$

It is evident that L_c is mapped to a horizontal segment and the discriminant curve reappears as the curve formed by the endpoints of these segments. This description of $\mathfrak{R}$ requieres a classical criterion for the roots of a cubic polynomial:

Cubic root criterion: the nature of the roots of $x^3 + ax + b = 0$ is determined by the discriminant $D = b^2/4 + a^3/27$. In detail

$$\text{Nature of roots} = \begin{cases} D > 0 & \rightarrow \text{ one real root, two complex conjugate} \\ D = 0 & \rightarrow \text{ three real roots at least two equal} \\ D < 0 & \rightarrow \text{ three real and distinct roots.} \end{cases}$$

Theorem 5.1. Let $c \in \mathbb{R}$ and define

$$a_c := \frac{c + 18}{4(c + 2)^{1/3}} \quad \text{and} \quad b_c := \frac{c + 4}{(c + 2)^{2/3}}. \tag{5.2}$$

The line L_c is mapped to the part of the horizontal line $b = b_c$ above the segment $(-\infty, a_c)$. The point (a_c, b_c) is on the discriminant curve $\mathfrak{R}$ and it provides a parametrization for it. For $-\infty < c \leq -2$ the image ends on the left branch $\mathfrak{R}_-$; for $-2 \leq c \leq 6$ the image ends on the left branch of the cusp piece $\mathfrak{R}_+$ and for $6 \leq c < +\infty$ it ends on the right cusp piece.

FIGURE 2. The range of Φ_6.

Proof. From the expression for Φ_6 it is clear that $\Phi_6(L_c)$ is contained in the horizontal line

$$b \quad \equiv \quad b_c := \frac{c+4}{(c+2)^{2/3}}, \tag{5.3}$$

and the part of this line that is actually achieved is parametrized by

$$a_1(c) \quad = \quad -\frac{a^2 - ac - (5c+9)}{(c+2)^{4/3}}.$$

We conclude that Φ_6 folds the line L_c over the horizontal segment $(-\infty, a_c)$ at height b_c.

The function b_c maps $\mathbb{R} - \{-2\}$ onto $\mathbb{R}$ and given a value of b, the parameter c is determined by the cubic equation

$$t^3 + (12 - b^3)t^2 + 48t + 64 \quad = \quad 0, \tag{5.4}$$

that normalizes to

$$t^3 + (8b^3 - b^6/3)t + (-16b^3 + 8b^6/3 - 2b^9/27) \quad = \quad 0 \tag{5.5}$$

of discriminant

$$D \quad = \quad -\frac{64}{27}(b-3)b^6(b^2+3b+9). \tag{5.6}$$

By the cubic root criterion 5.4) has exactly one real root for $b < 3$ and three distinct roots for $b > 3$. The proof is complete.

$\square$

Note. The points

$$a_c = \frac{c+18}{4(c+2)^{1/3}} \quad \text{and} \quad b_c = \frac{c+6}{(c+2)^{2/3}} \tag{5.7}$$

are on the discriminant curve $R(a,b) = 0$. They provide a parametrization of this curve. A simpler version of this parametrization is given by

$$a(s) = \frac{s^3+4}{s^2} \quad \text{and} \quad b(s) = \frac{s^3+16}{4s}. \tag{5.8}$$

The dynamics of (1.8) on the discriminant curve can be given explicitly in terms of (5.8). The next result can be verified directly.

Theorem 5.2. The function

$$\varphi(s) \quad = \quad \left(\frac{4(s^2+4)^2}{s(s+2)^2} \right)^{1/3} \tag{5.9}$$

satisfies

$$\Phi_6(a(s), b(s)) \quad = \quad (a(\varphi(s)), b(\varphi(s))) \,. \tag{5.10}$$

6. Fixed points

The goal of this section is to analyze the fixed points of the map Φ_6 defined in (1.7). These points satisfy

$$\frac{ab+5a+5b+9}{(a+b+2)^{4/3}} \quad = \quad a \tag{6.1}$$

$$\frac{a+b+6}{(a+b+2)^{2/3}} \quad = \quad b. \tag{6.2}$$

Lemma 6.1. There are three fixed points of Φ. They are given by

(1) $P_1 := (3,3)$
(2) $P_2 := (-3+r-2r^2, r^2-1) \approx (-4.205569, 3.957741)$ where $r \approx 1.353210$ is a root of z^3+z^2-z-2.
(3) $P_3 := (-3+r, 3+2r-r^2) \approx (-5.309144, 0.8311772)$ where $r \approx 1.205569$ is a root of z^3-z^2-z-2.

Moreover, $R(P_1)=0$, $R(P_2)=0$, and $R(P_3)<0$. The point P_1 is superattracting, P_2 is a saddle point, and P_3 is a stable spiral.

Proof. Introduce the auxiliary variable $z=(a+b+2)^{1/3}$. Then (6.2) yields

$$\begin{aligned} a &= z^3-2-z^{-2}(z^3+4) \\ b &= z^{-2}(z^3+4), \end{aligned}$$

and then (6.2) produces

$$\begin{aligned} P(z) &= z^{11}-z^9-3z^8-5z^7-3z^6-2z^5+z^4+8z^3+8z^2+16 \\ &= (z-2)(z^2+z-1)(z^2+z+2)(z^3+z^2-z-2)(z^3+z^2+z+2). \end{aligned}$$

The real roots of $P(z)=0$ are therefore $z_1=2$, yielding P_1, and two roots coming from the cubic factors that produce the other two fixed points. Direct calculation shows that two of the fixed points are on the resolvent curve.

The linearization Φ_6 at (a,b) is given by the matrix

$$J(a,b) \quad = \quad \begin{pmatrix} \frac{3b^2-ab+b-5a-6}{3(a+b+2)^{7/3}} & \frac{3a^2-ab+a-5b-6}{3(a+b+2)^{7/3}} \\ \frac{a+b-6}{3(a+b+2)^{5/3}} & \frac{a+b-6}{3(a+b+2)^{5/3}} \end{pmatrix} \tag{6.3}$$

of determinant

$$\operatorname{Det}(J(a,b)) = \frac{(b-a)(a+b-6)}{3(a+b+2)^3}. \tag{6.4}$$

This shows that P_1 is asymptotically stable, P_2 is a saddle point and P_3 is a spiral. This follows by a direct computation of the eigenvalues of $J(a,b)$ at each of these points. □

Lemma 6.2. The only 2-cycles above the line $a+b+2=0$ are fixed points.

Proof. Suppose $\Phi_6(a,b)=(c,d)$ and $\Phi_6(c,d)=(a,b)$. We need to show that $a=c$ and $b=d$. Letting $m=(a+b+2)^{1/3}$ and $n=(c+d+2)^{1/3}$ yields

$$\begin{aligned} a &= m^3 - 2 - \frac{n^3+4}{n^2} \\ b &= \frac{n^3+4}{n^2} \\ c &= n^3 - 2 - \frac{m^3+4}{m^2} \\ d &= \frac{m^3+4}{m^2} \end{aligned}$$

Substituting these into the first components of Φ_6, namely

$$\begin{aligned} a &= \frac{cd+5c+5d+9}{n^4} \\ c &= \frac{ab+5a+5b+9}{m^4}, \end{aligned}$$

yields two equations whose difference has the factor $n-m$ and the term

$$\begin{aligned} &n^4m(m^5-m^2+1)+m^4n(n^5-n^2+1)+mn((mn)^3-5(mn)^2+6mn+1) \\ &\quad +mn((m^2n^2-2)^2+m^2(2n^2-3n+2)+n^2(2m^2-3m+2)+3) \\ &\quad +8m+8n+2m^4++m^3+8m^2+n^3+8n^2+2n^4+m^5+n^5+n^2m+nm^2. \end{aligned}$$

The brackets indicate why this expression is positive when $m,n>0$. This forces $m=n$, hence $a=c$ and $b=d$. □

7. The region of convergence of the iteration

In section 3 we have shown that the map (3.1) converges to $(3,3)$ if the initial data (a_0,b_0) is in the first quadrant. In this section we describe the full region of convergence of the dynamical system (3.1).

It is easier to visualize this basin of attraction by sketching the set $\mathfrak{R} := \{(x,y) : R(x,y)=0\}$. We divide this curve into five pieces, $\{Z_k\}_{k=1}^5$, omitting the endpoints $(3,3)$, P_2 and $(-1,-1)$; see figure 7.

The main theorem claims that the basin of attraction of $(3,3)$ is the set of points above the part of the curve $R=0$ which contains Z_1, Z_2, Z_3, that file=plane.eps,width=25em,angle=0

FIGURE 3. The fixed points of Φ_6

is, the curve $\mathfrak{R}_- := Z_1 \cup P_2 \cup Z_2 \cup (-1,-1) \cup Z_3$. This region is denoted by Λ in Section 2.

The proof will require a technical lemma, that we prove next.

Lemma 7.1. The function

$$f(x,y,w) := \frac{(1+w)(x+y-6)(x+y+2)^{2/3}}{3(x+y+2)(xw+y+5+5w)-4(xy+5x+5y+9)(1+w)}$$

satisifes $-1 \leq f(x,y,w) \leq 0$ on the set D, where

$$D := \{(x,y,w) : -4.206 \leq x \leq 0,\ 3 \leq y \leq 3.96,\ -1 \leq w \leq 0\}$$

Proof. The numerator of f is always negative in D. Let

$$g(x,y,w) := 3(x+y+2)(xw+y+5+5w) - 4(xy+5x+5y+9)(1+w)$$

be the denominator of f. We first show that $g > 0$ on D. It suffices to consider g when $w=0$ and $w=-1$ since g is linear in w. On D, we have

$$g(x,y,0) = -6 - xy - 5x + 3y^2 + y > 0$$

and

$$g(x,y,-1) = -x^2 - 2x + y^2 + 2y = -(x+1)^2 + (y+1)^2 > 0.$$

Therefore, g is positive on D. We conclude that $f < 0$ on D.

The inequality $f \geq -1$ on D is equivalent to $h > 0$ on D, where

$$\begin{aligned} h(x,y,w) \;\; &:= \;\; (1+w)(x+y-6)(x+y+2)^{2/3} + 3(x+y+2)(xw+y+5+5w) \\ & \quad\;\; -4(xy+5x+5y+9)(1+w). \end{aligned}$$

Again, since h is linear in w, it suffices to prove this when $w=0$ and $w=-1$. We have

$$h(x,y,-1) = 3(x+y+2)(y-x) > 0$$

on D. Lastly,

$$h(x,y,0) = (x+y-6)(x+y+2)^{2/3} - 6 - xy - 5x + 3y^2 + y.$$

The first term is minimized when $a+b = 6/5$, so one easily obtains $h(x,y,0) > 0$. □

Define E as the bounded region

$$E := \{(a,b) : a \leq 0, b \geq 3, a+b \leq 5.6, R(a,b) > 0\}.$$

Lemma 7.1. *The function Φ_6 is injective on the region E.*

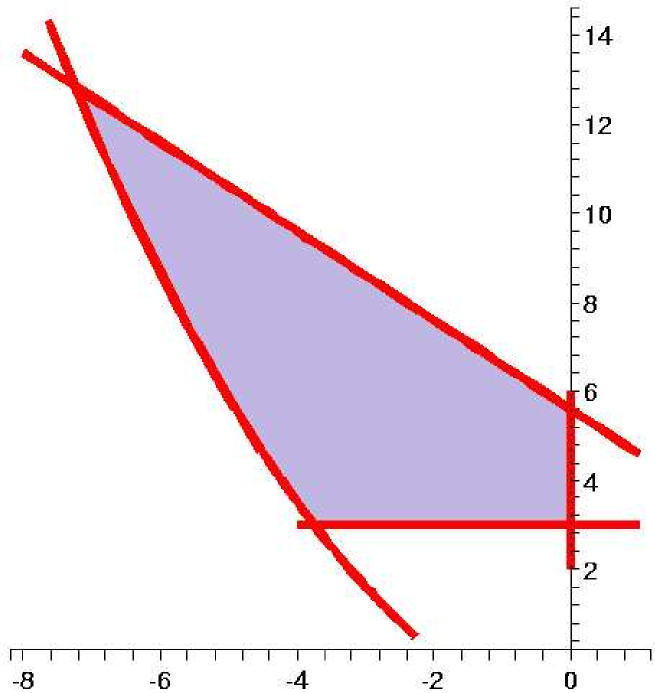

FIGURE 4. The region E.

Proof. Suppose $(a,b) \in E$, $(c,d) \in E$ and

$$\Phi_6(a,b) = \Phi_6(c,d). \tag{7.1}$$

Letting $u = (a+b+2)^(1/3)$ and $v = (c+d+2)^(1/3)$, the geometry of$E$ implies $u, v \in (1,2)$. The second component of Equation 7.1 yields

$$u + \frac{4}{u^2} = v + \frac{4}{v^2}.$$

Factoring forces either $u = v$ or $u^2v^2 - 4u - 4v = 0$. The latter case implies

$$\begin{aligned} u^2v^2 &= 4u + 4v \\ &\geq 8\sqrt{uv} \end{aligned}$$

hence $uv \geq 4$, contradicting that $u, v < 2$. We are then left with $u = v$. Equation 7.1 then produces

$$a + b = c + d, \ ab = cd.$$

This forces$ab = c(a + b - c)$, and solving for c, one finds $c = a$ or $c = b$. The domain E prohibits $c = b$, hence $c = a$, and subsequently $d = b$. This proves the desired result. $\square$

Now we give the proof of the main theorem.

Proof. The identity (1.10) indicates that the set $\mathfrak{R}_-$ is positively invariant and also tied to the line $L := \{(t,t) : t \neq -1\}$. Let L be divided into three pieces, $\{L_k\}_{k=1}^3$, omitting the endpoints $(-1,-1)$ and P_1. By choosing representives from each of these eight curves, one has the following observations about where these curves map:

(1) L_1 maps onto $Z_1 \cup \cup P_2 Z_2 \cup (-1,-1) \cup Z_3$.
(2) L_2 maps onto Z_4.
(3) L_3 maps onto Z_4.
(4) Z_1 maps into $Z_1 \cup P_2 \cup Z_2$.
(5) Z_2 maps onto Z_1.
(6) Z_3 maps onto $Z_1 \cup P_2 \cup Z_2$.
(7) Z_4 maps onto itself.
(8) Z_5 maps onto Z_4.

These relationships imply that the region Λ above $\mathfrak{R}_-$ is invariant and so is the region below this curve. The eigenvalues of the Jacobian of Φ_6 at P_1 are both zero, so this fixed point is not only asymptotically stable but the convergence is asymptotically quadratic. It remains to show that Λ is the basin of attraction for P_1.

The line $a+b+2=0$ lies below the curve $\mathfrak{R}_-$ except where it is tangent at $(-1,-1)$. This is readily seen since

$$R(a,-a-2) = -(a+1)^2(a^2+2a+5).$$

Therefore $a+b+2>0$ for all $(a,b) \in \Lambda$.

Note that the second coordinate of $\Phi_6(a,b)$ equals

$$\frac{a+b+6}{(a+b+2)^{2/3}} = w + \frac{4}{w^2} := f(w)$$

where $w = (a+b+2)^{1/3}$. When w is positive – which we know is true in Λ – the function f obtains a minimum at $w=2$ with $f(2)=3$. This means that Λ maps into the region $\tilde{\Lambda} := \Lambda \cap \{(a,b) : b \geq 3\}$, so it suffices to consider only this smaller region.

This argument may be sharpened further. Let $w_n = (a_n+b_n+2)^{1/3}$, where a_n and b_n are as in the statement of the theorem. Considering only $a_n \leq 0$ and $b_n \geq 3$ gives

$$\begin{aligned} w_{n+1}^3 &= a_{n+1}+b_{n+1}+2 \\ &\leq \frac{5w_n^3-1}{w_n^4} + \frac{w_n^3+4}{w_n^2} + 2 \end{aligned}$$

It is straightforward to show that if $w_n > 1.96$, then $w_{n+1} < w_n$, therefore (a_n,b_n) enters the first quadrant, or $w_n \leq 1.96$ eventually. This forces $a_n+b_n \leq 5.6$. We have reduced the problem to showing that all points in E eventually enter the first quadrant.

Since P_2 is a saddle point and the regions above and below the curve $\mathfrak{R}_-$ are invariant, this implies the stable manifold lies in $Z_1 \cup P_2 \cup Z_2$. Indeed, Lemma 6.2 implies this entire set is the stable manifold.

The positive eigenvalue at P_2 (~ 7.07) has a corresponding eigenvector whose slope is approximately -0.10367. In a neighborhood of P_2, the unstable manifold takes the form $(a, \phi(a))$ and its invariance implies the functional equation

$$\phi\left(\frac{a\phi(a) + 5a + 5\phi(a) + 9}{(a + \phi(a) + 2)^{4/3}}\right) = \frac{a + \phi(a) + 6}{(a + \phi(a) + 2)^{2/3}}$$

holds with $\phi'(-4.20557) \approx -0.10367$. Differentiating this equation gives

$$\phi'\left(\frac{a\phi(a) + 5a + 5\phi(a) + 9}{(a + \phi(a) + 2)^{4/3}}\right) =$$

$$\frac{(1 + w)(a + b - 6)(a + b + 2)^{2/3}}{3(a + b + 2)(aw + b + 5 + 5w) - 4(ab + 5a + 5b + 9)(1 + w)}$$

and this is the function $f(a, b, w)$ of Lemma 7.1, where $b = \phi(a)$ and $w = \phi'(a)$. Lemmas 7.1 and 6.1 imply that for $-4.206 < a < 0$ the unstable manifold $b = \phi(a)$ is a decreasing function which extends at least to $a = 0$. Therefore, the unstable manifold enters the positively invariant first quadrant. In Section 3 we have shown that if (a, b) is in the first quadrant, Φ_6 cuts the distance from (a, b) to P_1 by at least half, so the distance function acts as a Liapunov function for P_1 in this quadrant. Therefore, the unstable manifold from P_2 approaches P_1.

Lastly, consider the how the set E evolves. Since the portion on the boundary which lies on $R = 0$ approaches P_2, the parts of the boundary with $b = 3$ and $x + y = 5.6$ maps uniformaly close to the unstable manifold. Lemma **??** guarantees that the images of E stay between the images of the lower and upper boundaries, hence all points in E tend to P_1. This completes the proof.

$\square$

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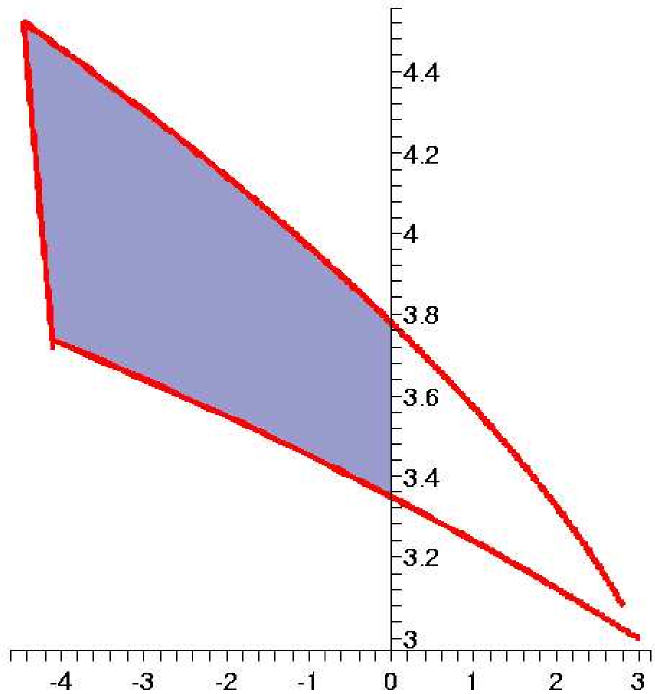

FIGURE 5. Three iterations of the region E.

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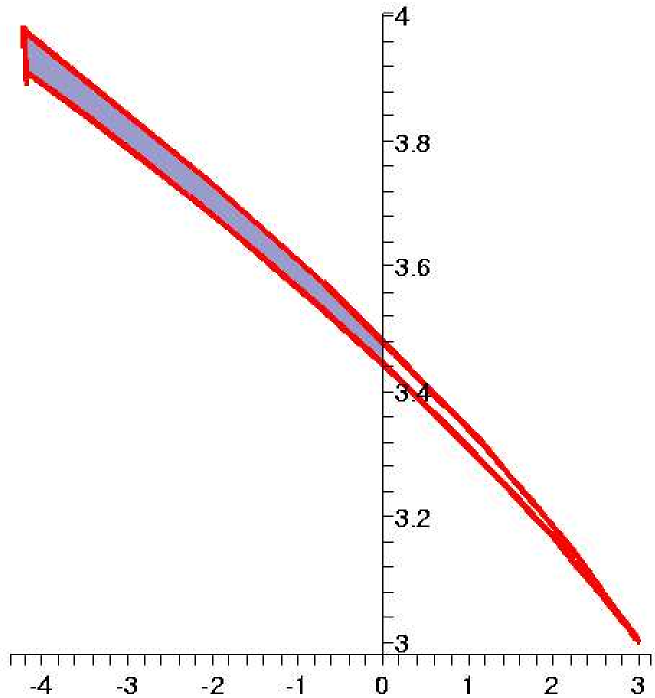

FIGURE 6. Six iterations of the region E.