A Generalization of the One-Seventh Ellipse

A curious mathematical phenomenon is called the *One-Seventh Ellipse* [1, 3]. Take the digits from the decimal expansion of $1/7$, namely 142857, and form six points in the plane: $(1,4)$, $(4,2)$, $(2,8)$, $(8,5)$, $(5,7)$ and $(7,1)$. The surprising fact is that these six points lie on an ellipse. Moreover, if we take consecutive digits from the decimal expansion two at a time, the six points $(14,28)$, $(42,85)$, $(28,57)$, $(85,71)$, $(57,14)$ and $(71,42)$ also lie on an ellipse. Figure 1 displays both ellipses. This note explains and generalizes these observations.

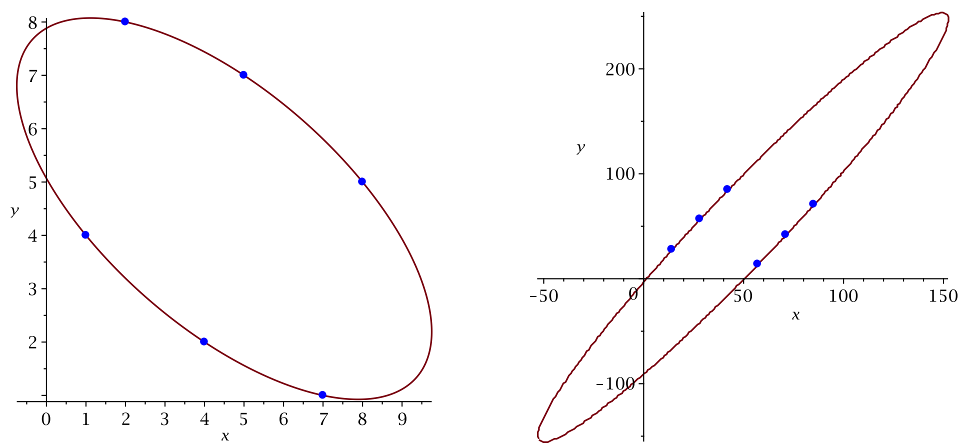

Figure 1: First and second ellipse.

It is natural to think that this phenomena is related to number theory. However, the tools we need come from geometry. From the set of points used in the one-seventh ellipse, notice that $1+8=4+5=2+7$, and also that $14+85=42+57=28+71$. More generally, we find that the sequence of six numbers

$$z_1 = a, z_2 = b, z_3 = c, z_4 = S - a, z_5 = S - b, z_6 = S - c$$

can be used to generate sets of six points in the plane (assuming $z_1, \ldots, z_6$ are distinct) that exhibit a beautiful structure. For each $n = 1, \ldots, 6$, define P_n as the set of six ordered pairs in the plane (z_i, z_{i+n}), $i = 1, \ldots, 6$, where wrap-around is used if necessary. When we refer to a conic section, we do not limit ourselves to the traditional curves obtained by slicing a cone with a plane, but rather, the locus of a quadratic polynomial in two variables. This includes ellipses, hyperbolas, a line, or a pair of parallel lines.

Theorem 1

Suppose that a, b, c, $S - a$, $S - b$, $S - c$ are six distinct real numbers. Then for each $n = 1, \ldots, 6$, we have

1. *The six points in P_n lie on a unique conic section. The center of each conic is located at $(S/2, S/2)$. For $n = 3$, the conic is the line $x+y = S$, while for $n = 6$, the conic is the line $x = y$.*

2. *If $n \neq n'$ are in $\{1, 2, 4, 5\}$, then $P_{n'}$ can be obtained by applying a reflection to P_n. The same reflection connects the conics associated with each sets of points.*

To prove the theorem, we will need some results from geometry. For context, we first state Pascal's Hexagrammum Mysticum Theorem (see Figure 2).

Proposition 2 (Pascal)

If a hexagon is inscribed in a conic, then the three points of intersection of the lines which contain opposite sides of the hexagon are collinear.

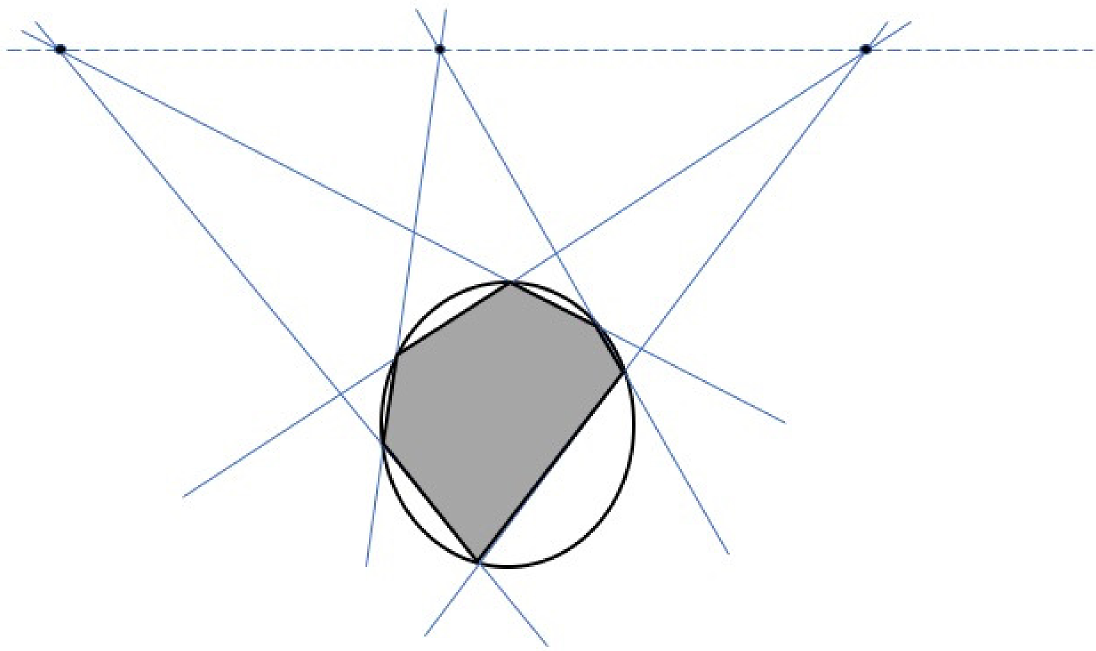

Figure 2: Pascal's Hexagrammum Mysticum Theorem.

Pascal's Theorem has an interesting converse.

Proposition 3 (Braikenridge-Maclaurin)

If three lines meet three other lines in nine points, and three of these points lie on a line, then the remaining six points lie on a conic.

Of interest here is that these theorems extend in the case of parallel lines when one adds a point at infinity. We are really working in the projective plane $\mathbb{R}P^2$. These results are described in Traves[2]. Using Proposition 3, we can now prove Theorem 1.

Proof. (Theorem 1)

To prove part (1), the cases $n = 3$ and $n = 6$ are trivial. Consider the set P_1. Form two sets of three lines:

$$\begin{aligned} S_1 \quad = \{ \quad & \text{line through } (a, b) \text{ and } (b, c), \\ & \text{line through } (c, S - a) \text{ and } (S - a, S - b), \\ & \text{line through } (S - b, S - c) \text{ and } (S - c, a). \ \} \end{aligned}$$

and

$$\begin{aligned} S_2 \quad = \{ \quad & \text{line through } (b, c) \text{ and } (c, S - a), \\ & \text{line through } (S - a, S - b) \text{ and } (S - b, S - c), \\ & \text{line through } (S - c, a) \text{ and } (a, b). \ \} \end{aligned}$$

By computing their slopes, one sees that each line in S_1 is parallel to a line in S_2. This implies that three of the nine points of interesection of S_1 and S_2 are the point at infinity, so these three points are collinear $\mathbb{R}P^2$. By the Braikenridge-Maclaurin Theorem, the remaining six points of intersection, namely the points in P_1, lie on a conic. A similar approach applies to the sets P_2, P_4, and P_5.

To prove part (2), we use reflections across the following lines: $y = x$, $x + y = S$, $x = S/2$ and $y = S/2$. One can show that P_j and P_k — $j, k \in \{1, 2, 4, 5\}$ and $j \neq k$ — are related by one or two of these reflections. For example,

$$P_1 = \{(a, b), (b, c), (c, S - a), (S - a, S - b), (S - b, S - c), (S - c, a)\}$$

can be reflected about the line $x = S/2$ to produce

$$P_4 = \{(S-a,b),(S-b,c),(S-c,S-a),(a,S-b),(b,S-a),(c,a)\}$$

We leave it to the reader to check the other connections.

To show that each conic section is also a result of the same reflections, observe that reflecting a conic section results in a conic section. Since six points determine at most one conic section, the set of reflections that take P_j to P_k will also take the conic through P_j to the conic through P_k.

□

Examples are straightforward to produce. To form an ellipse, take the original sequence $\{1,4,2,8,5,7\}$. To form a hyperbola, use the sequence $\{5,4,3,7,8,9\}$. To form a pair of parallel lines, take the sequence $\{1,2,4,8,7,5\}$. Figures 3, 4, and 5 show these three cases.

While Theorem 1 applies to any choice of $a,b,c \in \mathbb{R}$, suppose we try to connect this back to the original 1/7 observation and limit the parameters to $a,b,c,S-a,S-b,S-c \in \{0,1,2,3,4,5,6,7,8,9\}$. For which $x \in \mathbb{Q}$ will this produce a similar phenomenon? Of course this will produce only a finite number of possibilities. Writing the decimal expansion of x as an infinite

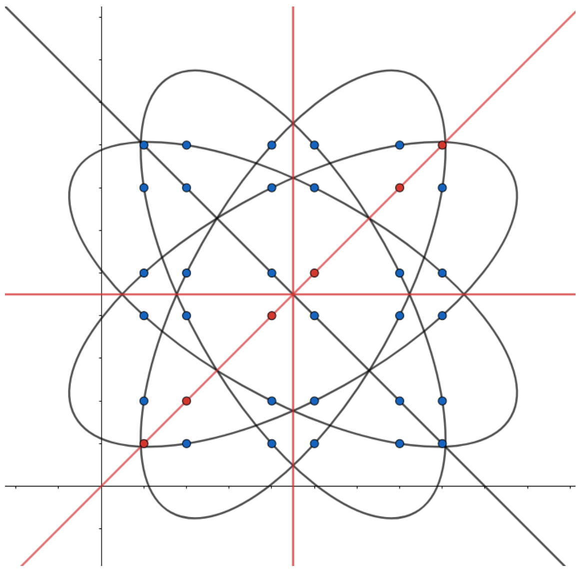

Figure 3: Ellipses.

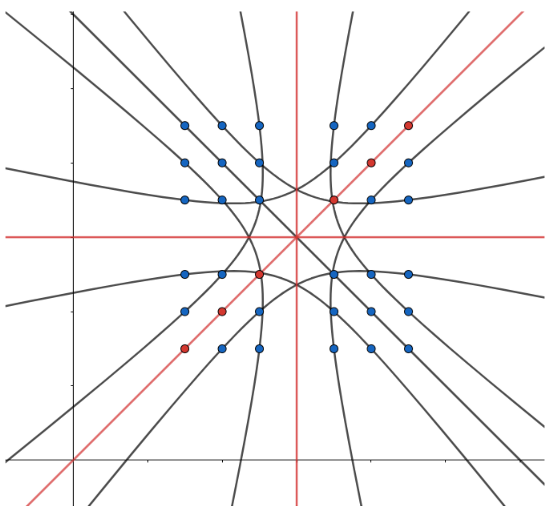

Figure 4: Hyperbolas.

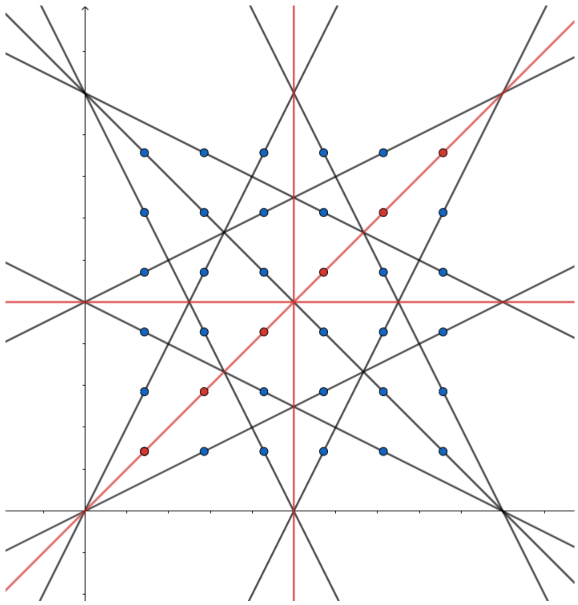

Figure 5: Parallel Lines.

series, we find

$$\begin{aligned} x &= \frac{a}{10}+\frac{b}{10^2}+\frac{c}{10^3}+\frac{S-a}{10^4}+\frac{S-b}{10^5}+\frac{S-c}{10^6}+\cdots \\ &= \left(\frac{a}{10}+\frac{b}{10^2}+\frac{c}{10^3}+\frac{S-a}{10^4}+\frac{S-b}{10^5}+\frac{S-c}{10^6}\right)\left(1+\frac{1}{10^6}+\frac{1}{10^{12}}+\cdots\right) \\ &= \left[\frac{999}{1000}\left(\frac{a}{10}+\frac{b}{10^2}+\frac{c}{10^3}\right)+\frac{S}{1000}\left(\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^3}\right)\right]\left(\frac{1}{1-\frac{1}{10^6}}\right) \\ &= [999(100a+10b+c)+111S]\,\frac{1}{10^6-1} \\ &= \frac{100a+10b+c}{1001}+\frac{S}{9009} \end{aligned}$$

For example, letting $a=1$, $b=2$, $c=5$ and $S=9$ produces the number $x=18/143$. One easily determines that the associated conics are ellipses.

References

[1] J. Hall. "One-Seventh Ellipse." From MathWorld–A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/One-SeventhEllipse.html

[2] W. Traves. *From Pascal's Theorem to d-Constructible Curves.* American Mathematical Monthly, 120(10), (2013), 901–915.

[3] D. Wells. *The Penguin Dictionary of Curious and Interesting Numbers.* Penguin Books, Middlesex, England, 1986.

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