A Natural Extension of the Pythagorean Equation to Higher Dimensions

Marc Chamberland
Department of Mathematics and Statistics
Grinnell College
Grinnell, Iowa 50112

August 25, 2008

Abstract.

The Pythagorean equation is extended to higher dimensions via circulant matrices. This form allows for the set of solutions to be expressed in a clean yet non-trivial way. The cubic case, namely the the equation $x^3 + y^3 + z^3 - 3xyz = 1$, was studied by Ramanujan and displays many interesting properties. The general case highlights the use of circulant matrices and systems of differential equations. The structure of the solutions also allows parametrized solutions of the Fermat equation in degrees 3 and 4 to be given in terms of theta functions.

AMS subject classification : 11D25, 11D41
Key Words. Pythagorean equation, Diophantine equations, circulant matrices, theta functions.
Running title: Generalized Pythagorean Equation

1 Introduction

It is well-known that the general integer solution to the Pythagorean equation

$$x^2 + y^2 = z^2 \tag{1}$$

is, allowing an interchange of x and y,

$$x = m^2 - n^2, \quad y = 2mn, \quad z = m^2 + n^2.$$

Properties of such triples, such as their connections to matrix generators and Fibonacci numbers, add to this subject's mystique; see Barbeau[1].

A natural extension to (1) is not obvious. It is now well-known that the Fermat equation $x^n + y^n = z^n$ with $n \geq 3$ has no solutions in positive integers. Other candidate equations, such as $x^2 + y^2 + z^2 = w^2$, have infinitely many solutions:

$$(m^2 - n^2 - p^2 + q^2)^2 + (2mn - 2pq)^2 + (2mp + 2nq)^2 = (m^2 + n^2 + p^2 + q^2)^2.$$

The equation $x^3 + y^3 + z^3 = w^3$ — or equivalently $x^3 + y^3 = z^3 + w^3$ — has a rich history (see Barbeau[1] and Dickson[7]). For example, Ramanujan found that if $a^2 + ab + b^2 = 3c^2d$, then

$$(a + cd^2)^3 + (bd + c)^3 = (ad + c)^3 + (b + cd^2)^3$$

Related are his "near misses" to the cubic Fermat equation[9, 10]: if

$$\begin{aligned}
\frac{1 + 53x + 9x^2}{1 - 82x - 82x^2 + x^3} &= \sum_{n=0}^{\infty} a_n x^n, \\
\frac{2 - 26x - 12x^2}{1 - 82x - 82x^2 + x^3} &= \sum_{n=0}^{\infty} b_n x^n, \\
\frac{2 + 8x - 10x^2}{1 - 82x - 82x^2 + x^3} &= \sum_{n=0}^{\infty} c_n x^n,
\end{aligned}$$

then

$$a_n^3 + b_n^3 = c_n^3 + (-1)^n.$$

Many other examples of Diophantine equations with infinitely many solutions are listed by Barbeau. Showing that a set of solutions is complete is usually much more challenging.

A natural extension of equation (1) should maintain properties of the original equation as well as its solutions. Two obvious properties of this equation are that it is *homogeneous* and all of its *infinitely many solutions* may be listed. The Fermat equation fails on the second count, however, a lesson can be learned here. A "naive approach"[14], extensively explored by Abel, Barlow, Germain and Legendre, is to factor the Fermat equation when n is odd as

$$\begin{aligned}
z^n &= x^n + y^n \\
&= (x + y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})
\end{aligned}$$

A Diophantine equation which admits a clear linear factorization and has the two desired properties is what is sought after here. This paper considers the equation

$$Q_n(x_1, x_2, \ldots, x_n) = s^n \tag{2}$$

where

$$Q_n(x_1, x_2, \ldots, x_n) \quad := \quad \det \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix}$$

Ramakrishnan[13] noted these determinants and their relation to the Lorentz transformation. In Ungar[16] and Muldoon and Ungar[12], the structure of circulant matrices is employed to offer natural extensions to the sin and cos functions. Some of this same structure is witnessed in this paper. Focusing on extending the Pythagorean equation, we shall see that equation (2) meets the desired criteria for all n. Section 2 lets us find the rational solutions in the case $n = 3$ before exploring the general case in Section 3. Section 4 considers other representations for solutions.

2 The Cubic Equation

The cubic version of equation (2) is

$$\begin{aligned} s^3 &= \det \begin{pmatrix} x & y & z \\ z & x & y \\ y & z & x \end{pmatrix} \\ &= x^3 + y^3 + z^3 - 3xyz \end{aligned}$$

The function $Q_3(x, y, z) = x^3 + y^3 + z^3 - 3xyz$ has arisen in various ways. Figure 1 displays the surface $Q_3(x, y, z) = 1$. Ramanujan[2, p.21] showed that if

$$a = x^2 + 2yz, \quad b = y^2 + 2zx, \quad c = z^2 + 2xy \tag{3}$$

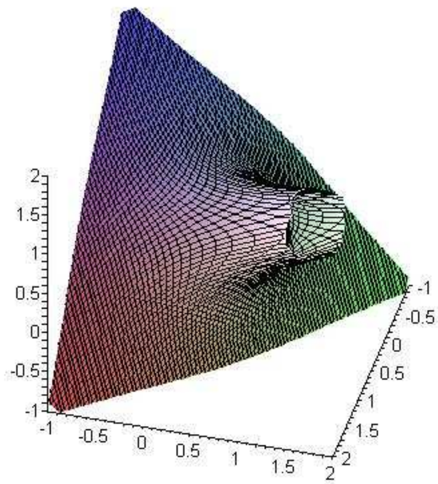

Figure 1: The surface $x^3 + y^3 + z^3 - 3xyz = 1$.

then

$$Q_3(a, b, c) = (Q_3(x, y, z))^2.$$

On a different note, define x, y, z as functions of t:

$$x = 1 + \frac{t^3}{3!} + \frac{t^6}{6!} + \cdots$$

$$y = t + \frac{t^4}{4!} + \frac{t^7}{7!} + \cdots$$

$$z = \frac{t^2}{2!} + \frac{t^5}{5!} + \frac{t^8}{8!} + \cdots$$

This implies[15, 16]

$$Q_3(x, y, z) = 1 \qquad (4)$$

for all t. The easy way to prove this uses the observation that $x' = z$, $y' = x$, and $z' = y$; the next section extends this for general n. A modified cubic version of the next section's main theorem may be expressed as follows.

Theorem 2.1 *The general rational solution of $Q_3(x,y,z)=s^3$, up to scaling, is generated by*

$$\begin{aligned} x &= a^3-c^3-t^3 \\ y &= b^3-a^3-t^3 \\ z &= c^3-b^3-t^3 \\ s &= \frac{-3t}{2}(a^2+b^2+c^2) \end{aligned}$$

where the rationals a,b,c,t satisfy the constraint $a+b+c=0$.

Gilman[8] has investigated this problem as well. Manipulating this formula leads to the following cute identity which is easily verifiable.

Corollary 2.1.1 *If $x+y+z=0$, then*

$$\left(4x^3-2y^3-2z^3\right)^2+\left(4y^3-2z^3-2x^3\right)^2+\left(4z^3-2x^3-2y^3\right)^2=9(x^2+y^2+z^2)^3$$

This result is similar to those of Ramanujan[2, p.96], later generalized by Bhargava[3].

3 General Case

The extension to n dimensions involves the polynomials Q_n defined as

$$Q_n(x_1,x_2,\ldots,x_n) := \det\begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix}$$

Since this is a circulant matrix, namely

$$Q_n(x_1,\ldots,x_n)=\det(circ(x_1,\ldots,x_n)),$$

its eigenvalues are known[17], giving

$$Q_n=\prod_{j=1}^{n}\sum_{k=1}^{n}x_k w^{kj}$$

where w is a primitive n^{th} root of unity. Note that Q_n is homogeneous of degree n.

To find all rational solutions to the equation $Q_n(x_1, x_2, \ldots, x_n) = s^n$, we begin with a lemma which highlights an important property of the polynomials Q_n. Throughout the following we define the auxiliary polynomial R_n as

$$\begin{aligned} R_n(x_1, \cdots, x_n) &:= \frac{Q_n(x_1, \cdots, x_n)}{x_1 + \cdots + x_n} \\ &= \prod_{j=1}^{n-1} \sum_{k=1}^{n} x_k w^{kj} \end{aligned}$$

Note that R_n is homogeneous of degree $n-1$.

Lemma 3.1 *For any a,*

$$Q_n(y_1 + a, \ldots, y_n + a) = Q_n(y_1, \ldots, y_n) + naR_n(y_1, \ldots, y_n)$$

Proof:

$$\begin{aligned} Q_n(y_1 + a, \ldots, y_n + a) &= (y_1 + \cdots + y_n + na)\ R_n(y_1 + a, \ldots, y_n + a) \\ &= (y_1 + \cdots + y_n + na)\ R_n(y_1, \ldots, y_n) \\ &= Q_n(y_1, \ldots, y_n) + naR_n(y_1, \ldots, y_n) \end{aligned}$$

The second equality makes repeated use of the identity $1 + w + \cdots + w^{n-1} = 0$ in each of the $n-1$ factors of R_n. □

For brevity, the notation Q_n or R_n without explicit arguments will always mean $Q_n(y_1, \cdots, y_n)$ or $R_n(y_1, \cdots, y_n)$. The main theorem now follows.

Theorem 3.1 *The general rational solution to $Q_n(x_1, \ldots, x_n) = s^n$ with $s \neq 0$ is given implicitly by*

$$Q_n\left(Q_n - t^n - nR_n y_1, \ldots, Q_n - t^n - nR_n y_n\right) = (-ntR_n)^n \qquad (5)$$

Proof: Let $(x_1, \ldots, x_n, s)$ be a different rational solution from $(1, \ldots, 1, 0)$. By considering the line joining $(x_1, \ldots, x_n, s)$ to $(1, \ldots, 1, 0)$ in $\mathbb{R}^{n+1}$, there exists integers $y_1, \ldots, y_n$ and t such that $y_k = (x_k - 1)t/s$. Since the two solutions differ, this forces $t \neq 0$, hence

$$x_k = 1 + \frac{y_k s}{t}.$$

Substituting this back into the Diophantine equation and using the lemma gives

$$\begin{aligned} s^n &= Q_n\left(1 + \frac{y_1 s}{t}, \ldots, 1 + \frac{y_n s}{t}\right) \\ &= Q_n\left(\frac{y_1 s}{t}, \ldots, \frac{y_n s}{t}\right) + nR_n\left(\frac{y_1 s}{t}, \ldots, \frac{y_n s}{t}\right) \\ &= \frac{s^n}{t^n} Q_n + n\frac{s^{n-1}}{t^{n-1}} R_n. \end{aligned}$$

Since $s \neq 0$, we have

$$s = \frac{-tnR_n}{Q_n - t^n}.$$

Substituting this back into the equation and scaling each term by $Q_n - t^n$ yields the desired result. □

The case $n = 3$ produces Theorem 2.1. To show this, first note that

$$\begin{aligned} R_3 &= y_1^2 + y_2^2 + y_3^2 - y_1 y_2 - y_1 y_3 - y_2 y_3 \\ &= \frac{1}{2}\left[(y_1 - y_2)^2 + (y_2 - y_3)^2 + (y_3 - y_1)^2\right] \end{aligned}$$

Letting $a = y_2 - y_1$, $b = y_3 - y_2$ and $c = y_1 - y_3$, we have

$$\begin{aligned} x_1 &= Q_3 - t^3 - 3R_3 y_1 \\ &= (y_2 + y_3 - 2y_1)R_3 - t^3 \\ &= (y_2 - y_1)^3 - (y_1 - y_3)^3 - t^3 \\ &= a^3 - c^3 - t^3 \end{aligned}$$

Similarly, $x_2 = b^3 - a^3 - t^3$ and $x_3 = c^3 - b^3 - t^3$. The constraint $a + b + c = 0$ clearly holds.

Some distinctive structure can be seen in the general solution if one of the x terms equals zero. Supposing $x_n = 0$, this forces $Q_n - t^n = nR_n y_n$, hence

$$Q_n\left(nR_n y_n - nR_n y_1, \ldots, nR_n y_n - nR_n y_{n-1}, 0\right) = (-ntR_n)^n$$

Since $s \neq 0$, this forces $R_n \neq 0$ and one may factor out $-nR_n$ to yield

$$Q_n\left(y_1 - y_n, \dots, y_{n-1} - y_n, 0\right) = t^n.$$

Lastly, equation (4) can be generalized as follows.

Theorem 3.2 *Suppose x_k, $k = 1, \dots, n$, are functions of t satisfying*

$$x_1' = x_2,\ x_2' = x_3,\ \ \dots,\ x_n' = x_1.$$

Then $Q_n(x_1, \dots, x_n)$ is a constant function in t.

Proof: For notational simplicity, let $x_{n+1} = x_1$. Then

$$\begin{aligned}
\frac{d}{dt} Q_n(x_1, x_2, \dots, x_n) &= Q_n \sum_{j=1}^{n} \frac{\frac{d}{dt}\left(\sum_{k=1}^{n} x_k w^{kj}\right)}{\sum_{k=1}^{n} x_k w^{kj}} \\
&= Q_n \sum_{j=1}^{n} \frac{\sum_{k=1}^{n} x_{k+1} w^{kj}}{\sum_{k=1}^{n} x_k w^{kj}} \\
&= Q_n \sum_{j=1}^{n} \frac{w^{-j} \sum_{k=1}^{n} x_{k+1} w^{(k+1)j}}{\sum_{k=1}^{n} x_k w^{kj}} \\
&= Q_n \sum_{j=1}^{n} w^{-j} \\
&= 0
\end{aligned}$$

□

The tie to infinite series is now immediate. Defining

$$x_k := \sum_{j=0}^{\infty} \frac{t^{k+jn}}{(k+jn)!},$$

the conditions of Theorem 3.2 are met. Evaluating the functions x_k at $t = 0$ lets us conclude $Q_n(x_1, \dots, x_n) = 1$ for all t. The special structure of this spacecurve may be witnessed yet another way. Following the approach of Mikusinski[11], one easily verifies that $x_k(t + s)$ satisfies the same differential equations. This implies that there are coefficients $\alpha_{k,1}, \dots, \alpha_{k,n}$ such that

$$x_k(t + s) = \alpha_{k,1} x_1(t) + \cdots + \alpha_{k,n} x_n(t)$$

where the coefficients depend only on s. Differentiating r times with respect to t then setting $t = 0$ gives

$$\alpha_{k,j}(s) = x_{k+j}(s)$$

where the subscript $k + j$ is taken modulo n to lie in $[1, n]$. Put together, we have

$$\begin{pmatrix} x_1(t+s) \\ x_2(t+s) \\ \vdots \\ x_n(t+s) \end{pmatrix} = \begin{pmatrix} x_n(s) & x_{n-1}(s) & x_{n-2}(s) & \cdots & x_1(s) \\ x_1(s) & x_n(s) & x_{n-1}(s) & \cdots & x_2(s) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_{n-1}(s) & x_{n-2}(s) & x_{n-3}(s) & \cdots & x_n(s) \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix},$$

This semigroup-type structure will reappear in the next section. Although the two-parameters on the right side suggest that a two-dimensional space of solutions is generated, the left side confirms that points starting on the curve stay there.

4 Real Solutions

While equation (5) was developed to generate rational solutions, one may replace the parameters with real values to generate real solutions to $Q_n(x_1, \ldots, x_n) = s^n$. Geometrically, this produces an $(n-1)$-dimensional surface in $\mathbb{R}^n$. When n is an odd prime, another nice representation for this surface is available.

Theorem 4.1 *Let p be an odd prime and $x_1, x_2, \ldots, x_p$ be implicitly defined as polynomials in $y_1, y_2, \ldots, y_p$ via the equation*

$$x_1 w + x_2 w^2 + \cdots + x_{p-1} w^{p-1} + x_p = (y_1 w + y_2 w^2 + \cdots + y_{p-1} w^{p-1} + y_p)^p \quad (6)$$

where w is a primitive p^{th} root of unity. Then

$$Q_p\left(\frac{x_1}{Q_p(y_1, y_2, \ldots, y_p)}, \frac{x_2}{Q_p(y_1, y_2, \ldots, y_p)}, \ldots, \frac{x_p}{Q_p(y_1, y_2, \ldots, y_p)}\right) = 1 \quad (7)$$

The special case $p = 3$ yields

$$x_1 = 3y_1^2 y_2 + 3y_2^2 y_3 + 3y_3^2 y_1,$$

$$x_2 = 3y_1y_2^2 + 3y_2y_3^2 + 3y_3y_1^2$$

$$x_3 = y_1^3 + y_2^3 + y_3^3 + 6y_1y_2y_3,$$

The theorem produces the identity

$$\begin{aligned}(y_1^3 + y_2^3 + y_3^3 + 6y_1y_2y_3)^3 + 27(y_1^2y_2 + y_2^2y_3 + y_3^2y_1)^3 + 27(y_1y_2^2 + y_2y_3^2 + y_3y_1^2)^3 \\ = 27(y_1^3 + y_2^3 + y_3^3 + 6y_1y_2y_3)(y_1^2y_2 + y_2^2y_3 + y_3^2y_1)(y_1y_2^2 + y_2y_3^2 + y_3y_1^2) \\ + (y_1^3 + y_2^3 + y_3^3 - 3y_1y_2y_3)^3\end{aligned}$$

Proof: The definition given in equation (6) holds for any primitive p^{th} root of unity. Since p is prime, this forces

$$x_1w^k + x_2w^{2k} + \cdots + x_{p-1}w^{k(p-1)} + x_p = (y_1w^k + y_2w^{2k} + \cdots + y_{p-1}w^{k(p-1)} + y_p)^p$$

for $k = 1, \ldots, p-1$. Note that the multinomial expansion implies

$$x_k = \sum_{\substack{j_1 + j_2 + \cdots + j_p = p \\ j_1 + 2\cdot j_2 + \cdots + p\cdot j_p \equiv k \bmod p \\ j_i \geq 0}} \begin{pmatrix} p \\ j_1 \; j_2 \; \cdots \; j_p \end{pmatrix} y_1^{j_1} y_2^{j_2} \cdots y_p^{j_p}$$

This may be used to obtain

$$\begin{aligned} x_1 + x_2 + \cdots + x_p &= \sum_{\substack{j_1 + j_2 + \cdots + j_p = p \\ j_i \geq 0}} \begin{pmatrix} p \\ j_1 \; j_2 \; \cdots \; j_p \end{pmatrix} y_1^{j_1} y_2^{j_2} \cdots y_p^{j_p} \\ &= (y_1 + y_2 + \cdots + y_p)^p \end{aligned}$$

Multiplying these equations together gives

$$\begin{aligned} Q_p(x_1, x_2, \cdots, x_p) &= \prod_{j=0}^{p-1} \sum_{k=1}^{p} x_k w^{kj} \\ &= \prod_{j=0}^{p-1} \left(\sum_{k=1}^{p} y_k w^{kj} \right)^p \\ &= Q_p(y_1, y_2, \cdots, y_p)^p \end{aligned}$$

This may be re-arranged to give the desired identity. □

Unlike equation (5), rational solutions of $Q_n(x_1, \ldots, x_n) = 1$ are not necessarily generated with rational y's in equation (7). In the $p = 3$ case, divide by y_1^3 and let $b = y_2/y_1$ and $c = y_3/y_1$. This implies the solution of $Q_3(x, y, z) = 1$ takes the form

$$x = \frac{3(b + b^2c + c^2)}{1 + b^3 + c^3 - 3bc}$$
$$y = \frac{3(b^2 + bc^2 + c)}{1 + b^3 + c^3 - 3bc}$$
$$z = \frac{1 + b^3 + c^3 + 6bc}{1 + b^3 + c^3 - 3bc}$$

Note that while $Q_3(2/3, 1, 4/3) = 1$, one finds the corresponding values of b and c to be

$$c = \beta, \quad b = -3\beta^2 + 26\beta - 7$$

where β is a real root of $Z^3 - 9Z^2 + 6Z - 1$. It is routine to show that β cannot be rational, therefore at least one of y_1 or y_3 is irrational.

Another important property of the polynomials Q_n may be seen as follows.

Theorem 4.2 *If $Q_n(x_1, \ldots, x_n) = s^n$, $Q_n(y_1, \ldots, y_n) = t^n$, and*

$$\begin{pmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 & \cdots & x_n \\ x_n & x_1 & x_2 & \cdots & x_{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ x_2 & x_3 & x_4 & \cdots & x_1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix},$$

then $Q_n(z_1, \ldots, z_n) = (st)^n$.

Proof:

$$\begin{aligned} Q_n(z_1, \ldots, z_n) &= \det(circ(z_1, \ldots, z_n)) \\ &= \det(circ(x_1, \ldots, x_n)) \det(circ(y_1, \ldots, y_n)) \\ &= Q_n(x_1, \ldots, x_n) Q_n(y_1, \ldots, y_n) \\ &= (s^n)(t^n) \\ &= (st)^n \end{aligned}$$

$\square$

This theorem may be used immediately to deduce Ramanujan's equation (3): let $n = 3$, $y_1 = x_1$, $y_2 = x_3$ and $y_3 = x_2$. Theorem 4.2 may also be used to give a new form for solutions to $Q_n(x_1, \ldots, x_n) = s^n$, $n = 3$ or 4, in terms of theta functions. In [5] and [6], it was shown that

$$a^3(q) = b^3(q) + c^3(q)$$

where $w = \exp(2\pi i/3)$, $-1 < q < 1$ and

$$a(q) := \sum_{m,n=-\infty}^{\infty} q^{m^2+mn+n^2},$$

$$b(q) := \sum_{m,n=-\infty}^{\infty} w^{m-n} q^{m^2+mn+n^2},$$

$$c(q) := \sum_{m,n=-\infty}^{\infty} q^{(m+1/3)^2+(m+1/3)(n+1/3)+(n+1/3)^2}.$$

Theorem 4.2 gives solutions to $Q_3(x_1, x_2, x_3) = 1$ as

$$\begin{aligned} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} &= \begin{pmatrix} b(p)/a(p) & c(p)/a(p) & 0 \\ 0 & b(p)/a(p) & c(p)/a(p) \\ c(p)/a(p) & 0 & b(p)/a(p) \end{pmatrix} \begin{pmatrix} b(q)/a(q) \\ c(q)/a(q) \\ 0 \end{pmatrix} \\ &= \frac{1}{a(p)a(q)} \begin{pmatrix} b(p)b(q) + c(p)c(q) \\ c(q)b(p) \\ b(q)c(p) \end{pmatrix} \end{aligned}$$

Similarly, the classical theta functions satisfy (see, for example, [4])

$$\theta_3^4(q) = \theta_4^4(q) + \theta_2^4(q)$$

where

$$\theta_2(q) := \sum_{n=-\infty}^{\infty} q^{(n+1/2)^2},$$

$$\theta_3(q) := \sum_{n=-\infty}^{\infty} q^{n^2},$$

$$\theta_4(q) := \sum_{n=-\infty}^{\infty} (-1)^n q^{n^2}.$$

Solutions to $Q_4(x_1, x_2, x_3, x_4) = 1$ may now be represented as

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{1}{\theta_3(p)\theta_3(q)\theta_3(r)} \begin{pmatrix} \theta_4(p) & \theta_2(p) & 0 & 0 \\ 0 & \theta_4(p) & \theta_2(p) & 0 \\ 0 & 0 & \theta_4(p) & \theta_2(p) \\ \theta_2(p) & 0 & 0 & \theta_4(p) \end{pmatrix}$$
$$\times \begin{pmatrix} \theta_4(q) & \theta_2(q) & 0 & 0 \\ 0 & \theta_4(q) & \theta_2(q) & 0 \\ 0 & 0 & \theta_4(q) & \theta_2(q) \\ \theta_2(q) & 0 & 0 & \theta_4(q) \end{pmatrix} \begin{pmatrix} \theta_4(r) \\ \theta_2(r) \\ 0 \\ 0 \end{pmatrix}$$

Acknowledgement: Many thanks are given to Jonathan Borwein, Karl Dilcher and Gary Walsh for helpful discussions and references. This work was conducted while on sabbatical leave at Dalhousie University with extra support from the Mellon Foundation.

References

[1] Barbeau, E. *Power Play.* Mathematical Association of America, (1997).

[2] Berndt, B. *Ramanujan's Notebooks, Part IV.* Springer-Verlag, New York, (1994).

[3] Bhargava, S. On a family of Ramanujan's formulas for sums of fourth powers. Ganita, 43(1992), 63–67.

[4] Borwein, J.M. and Borwein, P.B. *Pi and the AGM – A Study in Analytic Number Theory and Computational Complexity.* Wiley, New York, (1987).

[5] Borwein, J.M. and Borwein, P.B. A Cubic Counterpart of Jacobi's Identity and the AGM. Transactions of the American Mathematical Society, 323(1991), 691–701.

[6] Borwein, J.M., Borwein, P.B. and Garvan, F.G. Some Cubic Modular Identities of Ramanujan. Transactions of the American Mathematical Society, 343(1994), 35–47.

[7] Dickson, L.E. *History of the Theory of Numbers, Volume 2* . Chelsea, New York, (1952).

[8] Gilman, A.F. The Diophantine Equation $x^3 + y^3 + z^3 - 3xyz = p^r$. International Journal of Mathematical Education in Science and Technology, 28(1997), 468–472.

[9] Hirschhorn, M.D. An Amazing Identity of Ramanujan. Mathematics Magazine, 68(1995), 199–201.

[10] Hirschhorn, M.D. A Proof in the Spirit of Zeilberger of an Amazing Identity of Ramanujan. Mathematics Magazine, 69(1996), 267–269.

[11] Mikusinski, J.G. Sur les fonctions $k_n(x) = \sum_{\nu=0}^{\infty}(-1)^\nu(x^{n+k\nu}/(n + k\nu))!(k = 1, 2, \cdots; n = 0, 1, \cdots, k-1)$. (French) Ann. Soc. Polon. Math. ,21(1948), 46–51.

[12] Muldoon,M. and Ungar, A. Beyond Sin and Cos Mathematics Magazine, 69(1996), 3–14.

[13] Ramakrishnan, A. Cubic and General Extensions of the Lorentz Transformation. Journal of Mathematical Analysis and Applications, 229(1999), 88–92.

[14] Ribenboim, P. *13 Lectures on Fermat's Last Theorem*. Springer-Verlag, New York, (1979).

[15] Stewart, J. *Calculus,* 4^{th} *edition, p.814*. Brooks/Cole, (2001).

[16] Ungar, A. Generalized Hyperbolic Functions American Mathematical Monthly, 89(9)(1982), 688–691.

[17] E.W. Weisstein. *CRC Concise Encyclopedia of Mathematics*. Chapman and Hall/CRC, 1999.