Ramanujan's 6-8-10 Equation and Beyond

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Among Ramanujan's many beautiful formulas is the 6-8-10 equation

$$\begin{aligned}
&64[(a+b+c)^6+(b+c+d)^6-(c+d+a)^6-(d+a+b)^6+(a-d)^6-(b-c)^6] \\
&\quad\times[(a+b+c)^{10}+(b+c+d)^{10}-(c+d+a)^{10}-(d+a+b)^{10}+(a-d)^{10}-(b-c)^{10}] \\
&\quad=45[(a+b+c)^8+(b+c+d)^8-(c+d+a)^8-(d+a+b)^8+(a-d)^8-(b-c)^8]^2
\end{aligned}$$

when $ad = bc$. Berndt and Bhargava[2] cite this as "one of the most fascinating identities we have ever seen." Letting

$$f_m = (1+x+y)^m+(-x-y-xy)^m-(-y-xy-1)^m-(xy+1+x)^m+(-1+xy)^m-(-x+y)^m. \tag{1}$$

and $b = ax$, $c = ay$, $d = axy$, Ramanujan's equation may be compactly stated as

$$45f_8^2 = 64f_6f_{10}. \tag{2}$$

Proofs of equation (2) may be found in references [3], [5] and [6]. How Ramanujan found this identity (as with many of his results) remains shrouded in mystery. He also discovered $f_2 = 0$ and $f_4 = 0$. Only many decades later was a similar equation found by Hirschhorn, specifically

$$21f_5^2 = 25f_3f_7. \tag{3}$$

The goal of this note is to show how these identities and others may be found.

Berndt[1] showed that the polynomials f_m associated with (2) have nice factorizations:

$$\begin{aligned}
f_6 &= 3\,x\,(x-1)\,(2\,x+1)\,(x+2)\,(x+1)\,y\,(y-1)\,(2\,y+1)\,(y+2)\,(y+1)\,, \\
f_8 &= 8\,x\,(x-1)\,(2\,x+1)\,(x+2)\,(1+x)\left(x^2+x+1\right)
\end{aligned}$$

$$
\begin{aligned}
& \cdot\, y\,(y-1)\,(2\,y+1)\,(y+2)\,(y+1)\left(y^2+y+1\right), \\
f_{10} \;=\; & 15\,x\,(x-1)\,(2\,x+1)\,(x+2)\,(1+x)\left(x^2+x+1\right)^2 \\
& \cdot\, y\,(y-1)\,(2\,y+1)\,(y+2)\,(y+1)\left(y^2+y+1\right)^2.
\end{aligned}
$$

These factorizations are easily replicated by Maple, but one can prove these formally by showing that on each side the polynomials have the same zeros (with multiplicity) and scaling factor. Ramanujan's identity (2) now becomes straightforward to demonstrate. Similarly, the polynomials f_m in Hirschhorn's equation (3) may be factored to obtain

$$
\begin{aligned}
f_3 &= -3\,(x-1)\,(2\,x+1)\,(x+2)\;y\,(y+1), \\
f_5 &= -5\,(x-1)\,(2\,x+1)\,(x+2)\left(x^2+x+1\right)\,y\,(y+1)\left(y^2+y+1\right), \\
f_7 &= -7\,(x-1)\,(2\,x+1)\,(x+2)\left(x^2+x+1\right)^2\,y\,(y+1)\left(y^2+y+1\right)^2,
\end{aligned}
$$

allowing an easy proof of equation (3).

Are there other relationships among the polynomials f_m? Scanning the factors, one finds

$$5 f_3 f_8 = 8 f_5 f_6 \tag{4}$$

and

$$15 f_6 f_7 = 7 f_3 f_{10} \quad . \tag{5}$$

In an attempt to find other identities, one may factor f_m for other values of m. Unfortunately, the factorizations don't yield any obvious treasures. Maple produces, for example,

$$
\begin{aligned}
f_9 = & -3\,(x-1)\,(2\,x+1)\,(x+2)\;y\,(y+1)\,\big(3+27\,xy+9\,x+9\,y+21\,x^2+19\,y^2+63\,x^2y \\
& +57\,xy^2+27\,x^3+23\,y^3+105\,x^2y^2+81\,x^3y+115\,x^3y^2+69\,xy^3+105\,x^2y^3 \\
& +95\,x^3y^3+21\,x^4+19\,y^4+63\,x^4y+57\,xy^4+105\,x^4y^2+105\,x^2y^4+27\,x^5y \\
& +27\,xy^5+9\,x^5+9\,y^5+3\,x^6+3\,y^6+105\,x^4y^3+57\,x^5y^2+115\,x^3y^4+9\,x^6y \\
& +63\,x^2y^5+9\,xy^6+105\,x^4y^4+69\,x^5y^3+81\,x^3y^5+19\,x^6y^2+21\,x^2y^6+57\,x^5y^4 \\
& +63\,x^4y^5+27\,x^5y^5+23\,x^6y^3+27\,x^3y^6+19\,x^6y^4+21\,x^4y^6+9\,x^6y^5+9\,x^5y^6+3\,x^6y^6\big)
\end{aligned}
$$

and

$$f_{11} = -11\,(x-1)\,(2\,x+1)\,(x+2)\left(x^2+x+1\right)\,y\,(y+1)\left(y^2+y+1\right)(1+3\,x+3\,y+9\,xy$$
$$+9\,x^2+7\,y^2+27\,x^2y+21\,xy^2+39\,x^3y+33\,x^2y^2+27\,xy^3+13\,x^3+9\,y^3+9\,x^4+7\,y^4$$
$$+3\,x^5+3\,y^5+x^6+y^6+27\,x^4y+31\,x^3y^2+21\,x^2y^3+21\,xy^4+33\,x^4y^2-3\,x^3y^3$$
$$+33\,x^2y^4+9\,x^5y+9\,xy^5+21\,x^5y^2+21\,x^4y^3+31\,x^3y^4+3\,x^6y+27\,x^2y^5+3\,xy^6$$
$$+7\,x^6y^2+27\,x^5y^3+33\,x^4y^4+39\,x^3y^5+9\,x^2y^6+9\,x^6y^3+21\,x^5y^4+27\,x^4y^5+13\,x^3y^6$$
$$+7\,x^6y^4+9\,x^5y^5+3\,x^6y^5+9\,x^4y^6+3\,x^5y^6+x^6y^6\big)$$

By considering smaller values of m, one finds that f_{-1} and f_{-2} also have tidy factorizations, namely

$$f_{-1} = \frac{(x-1)\,(2\,x+1)\,(x+2)\,(x^2+x+1)\,y\,(y+1)\,(y^2+y+1)}{(1+x+y)\,(x+y+xy)\,(y+xy+1)\,(xy+1+x)\,(-1+xy)\,(x-y)}$$

and

$$f_{-2} = \frac{x\,(x-1)\,(2\,x+1)\,(x+2)\,(x+1)\,(x^2+x+1)^2\,y\,(y-1)\,(2\,y+1)\,(y+2)\,(y+1)\,(y^2+y+1)^2}{(1+x+y)^2\,(x+y+xy)^2\,(y+xy+1)^2\,(xy+1+x)^2\,(xy-1)^2\,(x-y)^2}$$

Not surprisingly, there are relationships between f_{-1}, f_{-2} and the other nicely factored terms:

$$f_{-2}f_3^2 = -3f_{-1}^2 f_6. \tag{6}$$

Just as with f_9 and f_{11}, f_m does not seem to factor extensively for integers $m \le -3$.

One may think this is the end of the road; however, note that the indices for each term in equation (2) sum to 16, 10 in equation (3), 11 in equation (4) and 13 in equation (5). Searching for a similar equation where each term involves two f_ms whose indices sum to 14, one could look for constants a, b, c, and d for which

$$af_3f_{11} + bf_5f_9 + cf_6f_8 + df_7^2 = 0.$$

Note that any other possible terms are vacuous since f_1, f_2 and f_4 are each identically zero. By evaluating this equation at four points (x, y) and using linear algebra, one arrives at the conjecture

$$245 f_3 f_{11} - 539 f_5 f_9 + 330 f_7^2 = 0. \tag{7}$$

How can one prove this equation is valid? Entering the expression on the left into Maple and simplifying produces zero. Alternatively, one may also use Hirschhorn's generating function approach[5]. Defining

$$a_1 = 1 + x + y, \quad b_1 = -x - y - xy, \quad c_1 = -1 + xy,$$

$$a_2 = -y - xy - 1, \quad b_2 = xy + 1 + x, \quad c_2 = -x + y,$$

$$q = (x^2 + x + 1)(y^2 + y + 1), \quad p_1 = a_1 b_1 c_1, \quad p_2 = a_2 b_2 c_2,$$

one finds

$$f_3 = 2(p_1 - p_2), \quad f_5 = 5q(p_1 - p_2), \quad f_6 = 3(p_1^2 - p_2^2), \quad f_7 = 7q^2(p_1 - p_2),$$

$$f_8 = 8q(p_1^2 - p_2^2), \quad f_9 = 3(p_1 - p_2)(p_1^2 + p_1 p_2 + p_2^2 + 3q^3),$$

$$f_{10} = 15q^2(p_1^2 - p_2^2), \quad f_{11} = 11q(p_1 - p_2)(p_1^2 + p_1 p_2 + p_2^2 + q^3).$$

While one now clearly obtains not only equations (2) and (3) but also equation (7), this approach has its limitations. Though the polynomials f_m may be expressed in a more compact form using p_1, p_2, and q, these representations of f_m will also become unwieldy for modest values of m.

Yet another approach to establish (7) involves difference equations. Since f_m is a linear combination of six m^{th} powers, f_m satisfies a sixth order difference equation whose characteristic equation has the six bases as its roots. With the factorizations noted earlier, one finds

$$\begin{aligned} 0 &= (r - 1 - x - y)(r + x + y + xy)(r + y + xy + 1)(r - xy - 1 - x)(r + 1 - xy)(r + x - y) \\ &= r^6 - 2(x^2 + x + 1)(y^2 + y + 1)r^4 \\ &\quad + x(x + 1)(y - 1)(2y + 1)(y - 2)r^3 + (x^2 + x + 1)^2(y^2 + y + 1)^2 r^2 \end{aligned}$$

$$
\begin{aligned}
& -x(x+1)(x^2+x+1)(y-1)(2y+1)(y-2)(y^2+y+1)r \\
& -(xy+1+y)(1+x+y)(xy+1+x)(xy+x+y)(x-y)(xy-1) \\
= \quad & r^6 - \frac{6}{5}\frac{f_5}{f_3}r^4 - \frac{f_6}{f_3}r^3 + \frac{9}{25}\frac{f_5^2}{f_3^2}r^2 + \frac{3}{5}\frac{f_5 f_6}{f_3^2}r + \frac{1}{5}\frac{f_5}{f_{-1}}
\end{aligned}
$$

thus yielding

$$
0 = f_m - \frac{6}{5}\frac{f_5}{f_3}f_{m-2} - \frac{f_6}{f_3}f_{m-3} + \frac{9}{25}\frac{f_5^2}{f_3^2}f_{m-4} + \frac{3}{5}\frac{f_5 f_6}{f_3^2}f_{m-5} + \frac{1}{5}\frac{f_5}{f_{-1}}f_{m-6} \quad (8)
$$

for all m. Specific choices of m in equation (8) give some known formulas: $m = 4$ produces equation (6), $m = 7$ gives Hirschhorn's equation (3), $m = 8$ gives equation (4), and $m = 10$ (with help from equations (3) and (4)) yields Ramanujan's equation (2). Indeed, equation (8) may be used recursively to generate many formulas. To obtain equation (7), take the $m = 11$ equation multiplied by f_3, the $m = 9$ equation multiplied by f_5, then subtract. This eliminates the f_{-1} terms and, combined with previously discovered identities, yields the desired result.

The linear algebra approach used to find equation (7) may be used to find many identities. Other equations found include

$$
\begin{aligned}
308 f_{10}^2 &= 525 f_8 f_{12} - 300 f_6 f_{14}, \\
1763580 f_{11}^2 &= 2735810 f_9 f_{13} - 1172490 f_7 f_{15} + 144837 f_5 f_{17} + 71995 f_2 f_{19}, \\
6395400 f_{14}^2 &= 10445820 f_{12} f_{16} - 5448212 f_{10} f_{18} + 1460151 f_8 f_{20} + 49980 f_6 f_{22}.
\end{aligned}
$$

These equations were mentioned in reference [4]. Upon further reflection, one realizes that limiting each term to the product of two f_ms is unnecessary; one may use partitions of integers to find even more possibilities. This produces more identities than we know what to do with. A small sampling includes

$$
-35\, f_3^4 - 945\, f_6^2 - 972\, f_5 f_7 + 1260\, f_3 f_9 = 0
$$

$$
-88\, f_3^3 f_5 - 1485\, f_6 f_8 - 1584\, f_5 f_9 + 2160\, f_3 f_{11} = 0
$$

$$
3375\, f_3 f_6^2 - 4500\, f_3^2 f_9 + 2916\, f_5^3 + 125\, f_3^5 = 0
$$

$$7776\, f_5^2 f_7 + 4725\, f_3 f_6 f_8 - 10080\, f_3 f_5 f_9 + 280\, f_3^4 f_5 = 0$$

$$-35\, f_3^3 f_8 - 630\, f_8 f_9 - 108\, f_7 f_{10} + 540\, f_3 f_{14} = 0$$

$$-35\, f_3^3 f_8 - 630\, f_8 f_9 - 1296\, f_7 f_{10} + 1512\, f_5 f_{12} = 0$$

$$-2187\, f_5^2 f_8 - 1800\, f_3 f_6 f_9 - 100\, f_3^4 f_6 + 2700\, f_3^2 f_{12} = 0$$

A word should be said about the Maple code used to produce these examples. Maple's built-in partition capabilities make the code relatively short. However, since the partition function grows very quickly, even Maple's power will get bogged down after some time. For example, there are 627 partitions of the number 20. To reduce the amount of computing, recall that $f_m = 0$ for $m = 1, 2, 4$. This reduces the number of relevant partitions to 27, a much more manageable number. Lastly, each identity is factored to weed out those which are a multiplicative combination of others.

As a final exploration, one may discover combinations involving f_m with negative m:

$$12\, f_5 f_{-1} f_{-2} - 5\, f_6 f_{-1}^4 + 5\, f_6 f_{-2}^2 = 0$$

$$18\, f_5 f_{-1} f_{-2} - 5\, f_6 f_{-1}^4 + 5\, f_3^2 f_{-4} + 20\, f_6 f_{-1} f_{-3} = 0$$

$$-42\, f_5 f_{-1} f_{-2} + 5\, f_7 f_{-1}^3 f_{-2} - 15\, f_7 f_{-1} f_{-4} f_7 f_{-2} f_{-3} = 0$$

$$-36\, f_5 f_{-1}^3 - 5\, f_3^2 f_{-1}^4 + 5\, f_3^2 f_{-2}^2 = 0$$

$$-168\, f_5 f_{-2} - 36\, f_7 f_{-1}^2 f_{-2} + 280\, f_6 f_{-1}^3 + 21\, f_8 f_{-5} - 21\, f_8 f_{-1} f_{-2}^2 + 60\, f_7 f_{-4} = 0$$

$$45\, f_7 f_{-1}^4 + 700\, f_3^2 f_{-1}^3 + 84\, f_3 f_5 f_{-5} - 84\, f_3 f_5 f_{-1} f_{-2}^2 - 225\, f_7 f_{-2}^2 + 3780\, f_5 f_{-1}^2 = 0$$

$$1440\, f_7 f_{-1}^4 + 5600\, f_3^2 f_{-1}^3 + 2688\, f_3 f_5 f_{-5} - 168\, f_3 f_5 f_{-1} f_{-2}^2 + 3600\, f_7 f_{-2}^2 + 4725\, f_8 f_{-1} f_{-4} = 0$$

$$360\, f_7 f_{-1}^4 + 1400\, f_3^2 f_{-1}^3 + 672\, f_3 f_5 f_{-5} + 168\, f_3 f_5 f_{-1} f_{-2}^2 + 3600\, f_7 f_{-2}^2 + 1575\, f_8 f_{-2} f_{-3} = 0$$

$$120\, f_7 f_{-1}^2 f_{-2} - 840\, f_6 f_{-1}^3 + 105\, f_8 f_{-1} f_{-2}^2 + 56 f_3 f_5 f_{-2} f_{-3} = 0$$

$$-135\, f_7 f_{-1}^4 - 350\, f_3^2 f_{-1}^3 - 42\, f_3 f_5 f_{-5} + 42\, f_3 f_5 f_{-1} f_{-2}^2 - 225\, f_7 f_{-2}^2 + 450\, f_7 f_{-1} f_{-3} = 0$$

$$27\, f_8 f_{-1}^2 f_{-2} - 16\, f_3 f_6 f_{-2} f_{-3} + 4\, f_3 f_6 f_{-1}^3 f_{-2} + 12\, f_3 f_6 f_{-1} f_{-4} = 0$$

$$112\, f_3 f_5 f_{-1} f_{-3} - 105\, f_8 f_{-1}^2 f_{-2} + 1680\, f_6 f_{-2} + 960\, f_7 f_{-3} + 315\, f_8 f_{-4} = 0$$

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