When Are All the Zeros of a Polynomial Real and Distinct?

Marc Chamberland

Abstract. This note gives necessary and sufficient conditions for all the zeros of a single-variable polynomial with real coefficients to be real and distinct.

There are many results that give information about the zeros of single-variable polynomials with real coefficients. The theorems of Descartes, Fourier–Budan, and Sturm—and other results—can be found in [**1**] and a careful study of the geometry of a polynomial's zeros is found in the classic book by Marden [**6**]. This note constructs polynomial inequalities that are necessary and sufficient for all the zeros to be real and distinct. The proof is elementary and the inequalities depend only on the polynomial and its derivatives.

Let the expression $P^{(j)}$ denote the jth derivative of P. The main result follows.

Theorem 1. *Let P be a polynomial of degree $n \geq 1$ with real coefficients. Then the zeros of P are real and distinct if and only if*

$$(P^{(j)}(x))^2 - P^{(j-1)}(x)\, P^{(j+1)}(x) > 0 \tag{1}$$

for all $x \in \mathbb{R}$, $j = 1, 2, \ldots, (n-1)$.

The two directions of the proof will be settled separately. We first prove that the conditions are necessary.

Proof. (Theorem 1, ⇒)

Write P as

$$P(x) = C(x - r_1)(x - r_2) \cdots (x - r_n)$$

for some real $C \neq 0$. By expanding the expression $(P'/P)'$, one finds that

$$(P'(x))^2 - P(x)P''(x) = (P(x))^2 \left[\frac{1}{(x-r_1)^2} + \frac{1}{(x-r_2)^2} + \cdots + \frac{1}{(x-r_n)^2}\right]$$

for all x except the zeros of P. Since the zeros are real, the right side is clearly positive for real x except possibly at the zeros. Since the zeros are distinct, we have

$$\lim_{x \to r_k} (P(x))^2 \left[\frac{1}{(x-r_1)^2} + \frac{1}{(x-r_2)^2} + \cdots + \frac{1}{(x-r_n)^2}\right] = C^2 \prod_{\substack{i=1 \\ i \neq k}}^{n} (r_k - r_i)^2 > 0$$

for $k = 1, 2, \ldots, n$. This forces

$$(P'(x))^2 - P(x)P''(x) > 0$$

for all $x \in \mathbb{R}$, thus settling the $j = 1$ case of the theorem. By using Rolle's theorem inductively, one sees that $P^{(j)}$ has $n - j$ distinct, real zeros, so the preceding analysis establishes the claim for $j = 2, 3, \ldots, (n-1)$.

■

This proof is essentially due to Laguerre. Indeed, Laguerre used these inequalities to study a broader class of entire functions of genus zero and genus one now commonly known as the *Laguerre–Pólya class* [**8**]. The inequalities (1) are usually referred to as the *Laguerre inequalities*.

A similar necessary condition to (1) known as *Newton's inequality* relates any three consecutive coefficients in the polynomial; see [**4, 9**]. Translating that expression in terms of derivatives of the polynomial, Newton's inequalities can be written as

$$(P^{(j)}(x))^2 - P^{(j-1)}(x)\, P^{(j+1)}(x) \left(1 + \frac{1}{n-j}\right) > 0$$

for all $x \in \mathbb{R}$, $j = 1, 2, \ldots, (n-1)$.

Note that if one only assumes that the zeros are real, the relation in (1) is replaced by "$\geq$." This result can be found as an exercise in [**2**, E.5c, p. 22] and related to a property of the Schwarzian derivative [**3**, p. 70]. While it is tempting to try to relax the conditions in Theorem 1, care must be taken. The polynomial $p(x) = x^4 - 1$ has zeros $x = \pm 1, \pm i$, but satisfies

$$(p^{(j)}(x))^2 - p^{(j-1)}(x)\, p^{(j+1)}(x) \geq 0$$

for all $x \in \mathbb{R}$, $j = 1, 2, 3$.

To support a proof of the other direction of Theorem 1, we start with a lemma.

Lemma 1. *Let Q be a polynomial that satisfies*

$$(Q'(x))^2 - Q(x)Q''(x) > 0 \tag{2}$$

for all $x \in \mathbb{R}$. Then Q has at least one real zero, all its real zeros are distinct, Q' has one fewer real zero than Q, and the real zeros of Q and Q' interlace.

Proof. Inequality (2) implies that Q is nonconstant and that any critical point of Q must be either a positive local maximum or a negative local minimum; hence Q must have a real zero. Inequality (2) also implies that Q's real zeros must be distinct, otherwise $(Q')^2 - QQ'' = 0$ at a repeated zero. Let $a_1 < a_2 < \cdots < a_k$ denote the real zeros of Q. By Rolle's theorem, between each pair of adjacent zeros lies a real zero b of Q', and the earlier observation about critical points implies that b is unique. Moreover, there cannot be a real zero b of Q' with $b > a_k$ since this would force Q to have a positive local minimum or negative local maximum—impossible as discussed earlier—or else Q has a horizontal asymptote, impossible for a nonconstant polynomial. The same argument implies that there cannot be a real zero b of Q' with $b < a_1$. ∎

We now prove that the conditions in Theorem 1 are sufficient.

Proof. (Theorem 1, $\Leftarrow$)

We argue by induction on the degree n. The base case, $n = 1$, trivially holds. Now assume that the theorem holds for a certain natural number n. If P is a polynomial of degree $n+1$ that satisfies (1) for $j = 1, 2, \ldots, n$, then P' satisfies (1) for $j = 1, 2, \ldots, (n-1)$, so by the inductive hypothesis, P' has n distinct, real zeros. Applying the lemma to the function P, we have that P has $n+1$ distinct, real zeros. ∎

The sufficiency condition (1) can be weakened by replacing it with the following:

$$P^{(j-1)}(\xi)P^{(j+1)}(\xi) < 0 \;\text{ if } P^{(j)}(\xi) = 0, \;\text{ where } 1 \leq j \leq n-1 \text{ and } \xi \in \mathbb{R}. \tag{3}$$

A careful examination of the lemma's proof shows that replacing (2) with (3)—applied to Q with $j = 1$— suffices to prove the lemma if one assumes that Q is a nonconstant polynomial. The sufficiency proof requires no alteration. This alternate sufficiency theorem using (3) is not new; it was highlighted by Pólya [**7**] who credits the result to Jean Paul de Gua de Malves in 1741. Of course, whether one chooses to use condition (1) or condition (3) depends on whether one wants to use necessary conditions or sufficient conditions. Another sufficiency condition similar to Newton's inequality is given in [**5**].

Finally, when a polynomial's zeros are not real and distinct, it is not clear which conditions from Theorem 1 fail. The polynomial $p(x) = x^3 + x$ has one real and two complex zeros, while (1) is satisfied for $j = 1$ and violated for $j = 2$. In contrast, the polynomial $p(x) = x^3 - x + 1$ has one real and two complex zeros, while (1) is satisfied for $j = 2$ and violated for $j = 1$.

ACKNOWLEDGMENT. The author is indebted to the anonymous referee for many helpful suggestions.

REFERENCES

1. Barbeau, E. (1989). *Polynomials*. New York: Springer.
2. Borwein, P., Erdélyi, T. (1995). *Polynomials and Polynomial Inequalities*. New York: Springer.
3. Elaydi, S. (2007). *Discrete Chaos*, 2nd ed. New York: Chapman & Hall/CRC.
4. Hardy, G. H., Littlewood, J., Pólya, G. (1967). *Inequalities*, 2nd ed. London: Cambridge Univ. Press.
5. Kurtz, D. C. (1992). A sufficient condition for all the roots of a polynomial to be real. *Amer. Math. Monthly*. 99(3): 259–263.
6. Marden, M. (1966). *Geometry of Polynomials*, 2nd ed. Providence, RI: American Mathematical Society.
7. Pólya, G. (1930). Some problems connected with Fourier's work on transcendental equations. *Quarterly J. Math.* 1(1): 21–34. doi.org/10.1093/qmath/os-1.1.21
8. Skovgaard, H. (1954). On inequalities of the Turán type. *Math. Scand.* 2: 65–73.
9. Wagner, C. G. (1977). Newton's inequality and a test for Imaginary Roots. *Two-Year College Math. J.* 8(3): 145–147.

Grinnell College, Grinnell, IA 50112
chamberl@grinnell.edu